Tilings by polygons

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Tilings

What shapes can tile?

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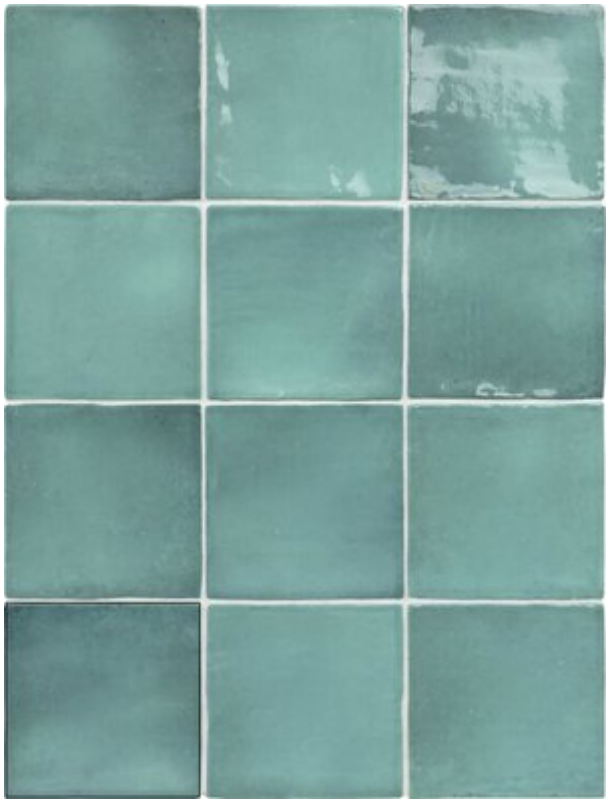


Part I: What polygons can tile?

A polygon is a shape with straight sides.

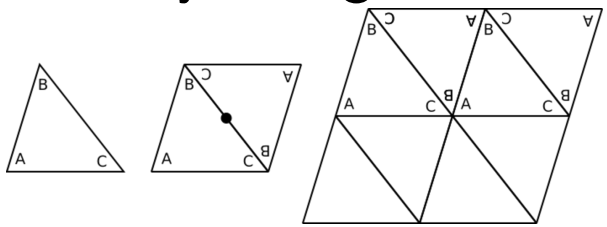
Can triangles tile?

A triangle is a polygon with three sides.

Can triangles tile?

A triangle is a polygon with three sides.

Yes! Any triangle can tile.



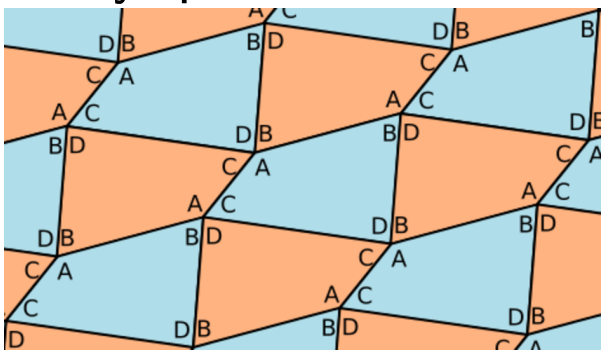
Can quadrilaterals tile?

A quadrilateral is a polygon with four sides.

Can quadrilaterals tile?

A quadrilateral is a polygon with four sides.

Yes! Any quadrilateral can tile.



Can pentagons tile?

A pentagon is a polygon with five sides.

Can pentagons tile?

A pentagon is a polygon with five sides.

Some can, some cannot.

Which pentagons can tile?

Which **convex** pentagons can tile?

Which pentagons can tile?

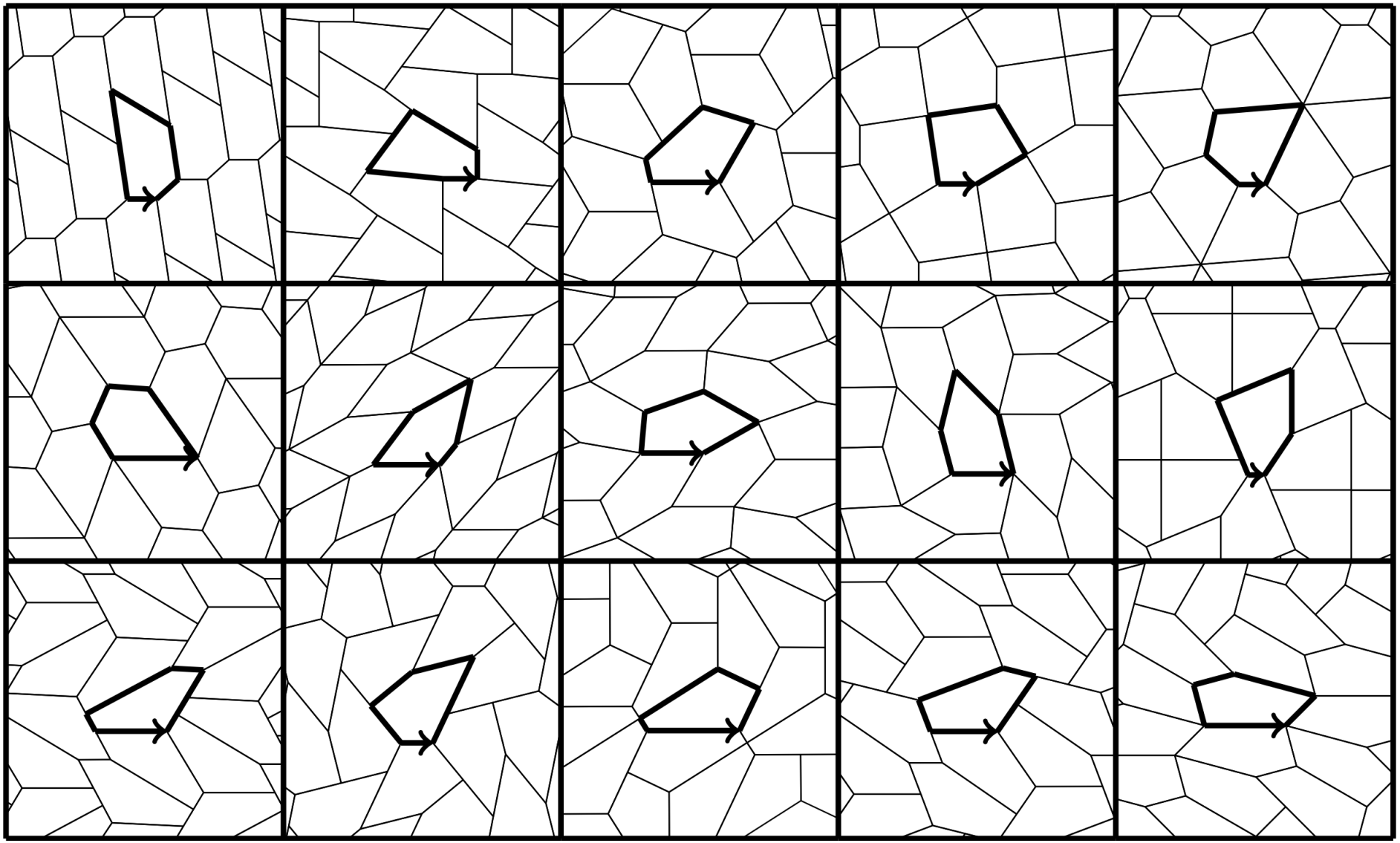
Marjorie Rice

Which **convex** pentagons can tile?

- 1918: Reinhard - five types
- 1968: Kershner - three more types
- 1975: James - one more type
- 1977: Rice - four more types
- 1985: Stein - one more
- 2015: Mann, McLoud and Von Derau - one more

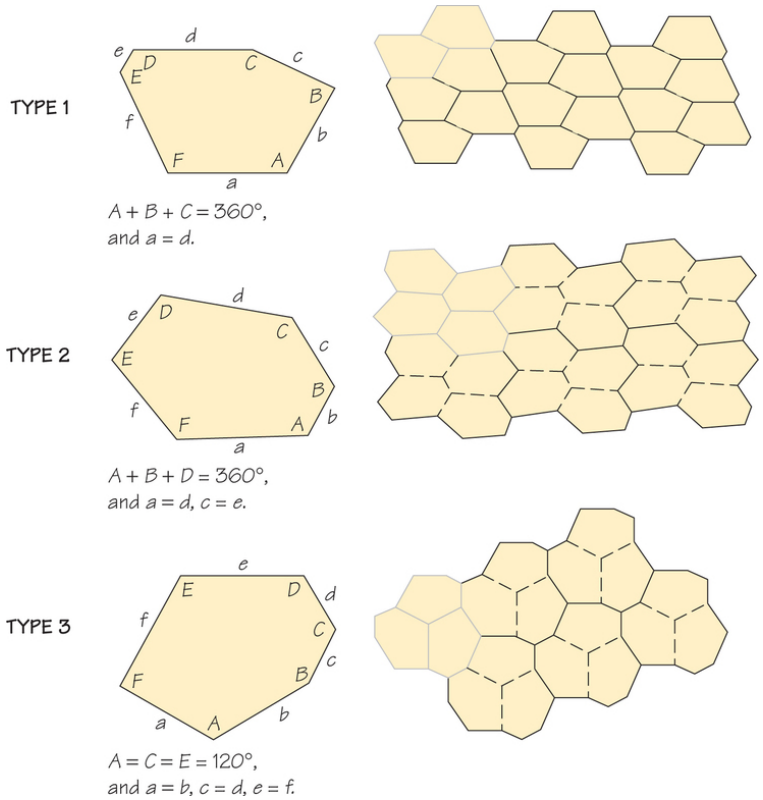
2017: Rao has a computer proof that these 15 are the only ones.

15 types of convex pentagons that tile



Convex polygons

- All triangles can tile.
- All quadrilaterals can tile.
- There are 15 types of convex pentagon that can tile. (2017)
- There are three types of convex hexagon that can tile. (1918)
- No convex tile with 7 or more sides can tile.

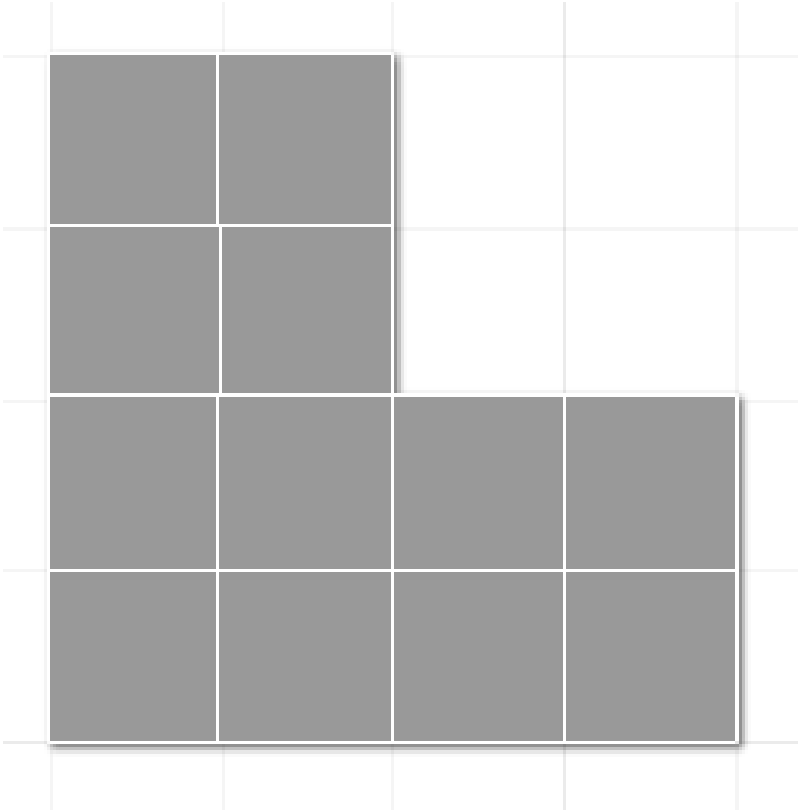


Part II: Aperiodic tilings

A tiling is **periodic** if the pattern repeats side-to-side and up-and-down.

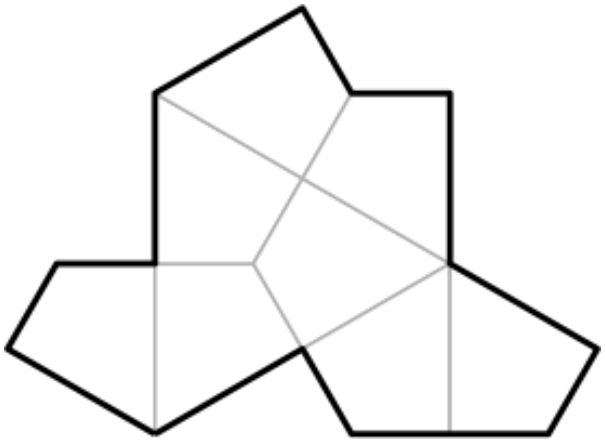
Can you design a tiling that is **not** periodic?

A rep-tile



The hat monotile

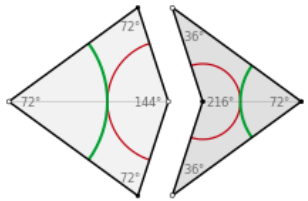
Can this shape tile?



Aperiodic monotiles

The "hat" tile is the first known aperiodic monotile.
Monotile : This one shape can tile the plane.
Aperiodic : It cannot tile with a repeating pattern.

- First set of aperiodic tiles: Wang, 1961.
- Penrose set of two tiles, 1972.



- "Hat" tile, discovered by David Smith, 2023.

Proofs?

Two key questions:

- How do we know the hat tile can actually tile the plane?
- How do we know it cannot have a repeating pattern?

Both proved in 2023 by David Smith, Joseph Samuel Myers, and Craig S. Kaplan, Chaim Goodman-Strauss.

Metatiles

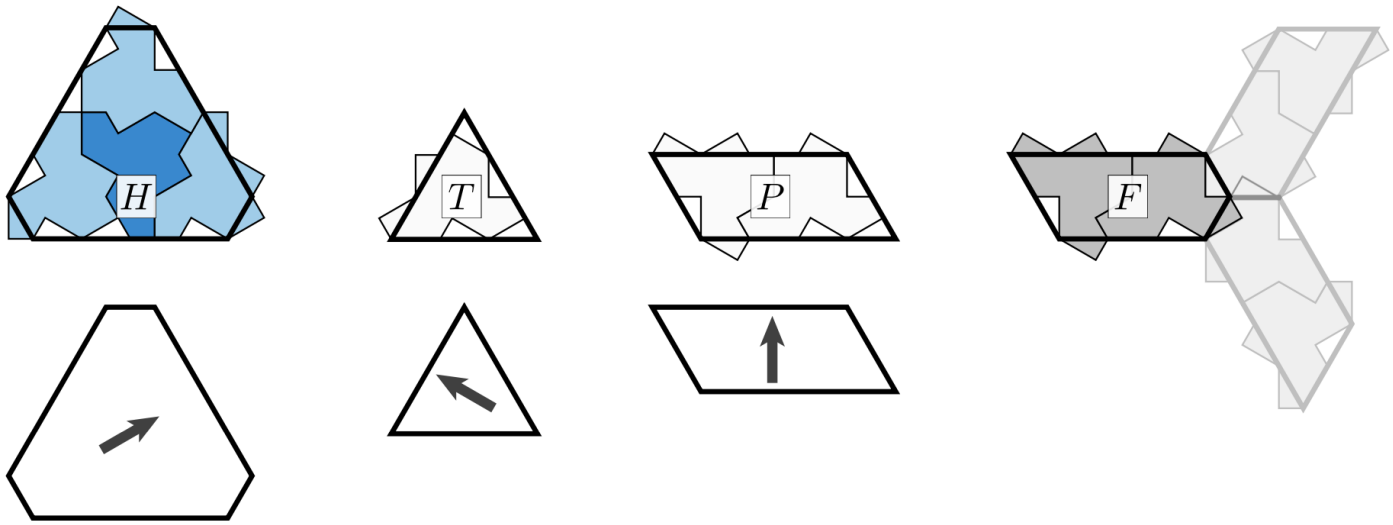
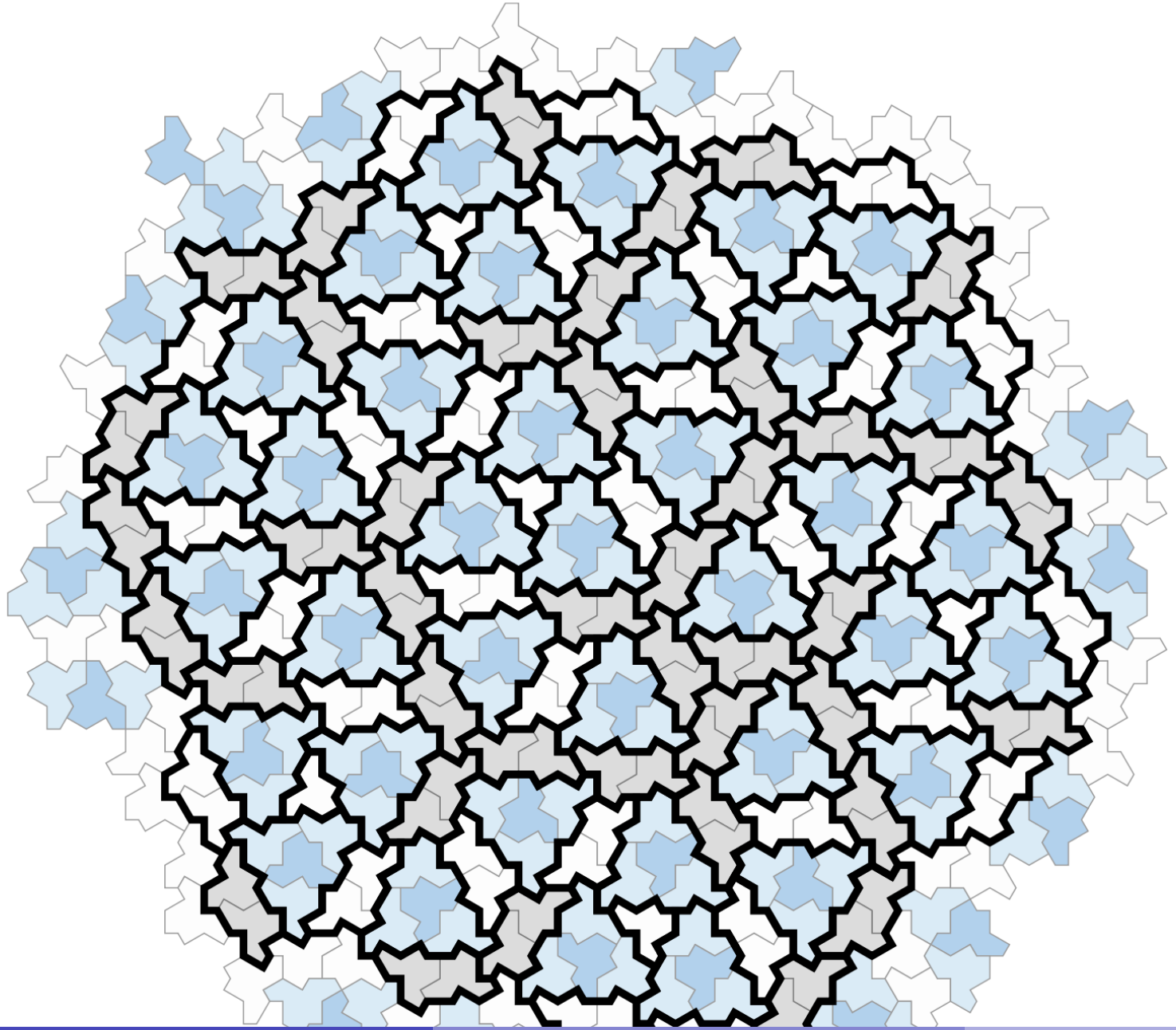
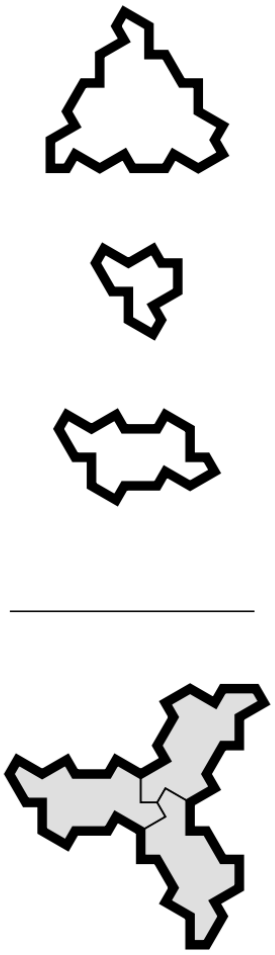


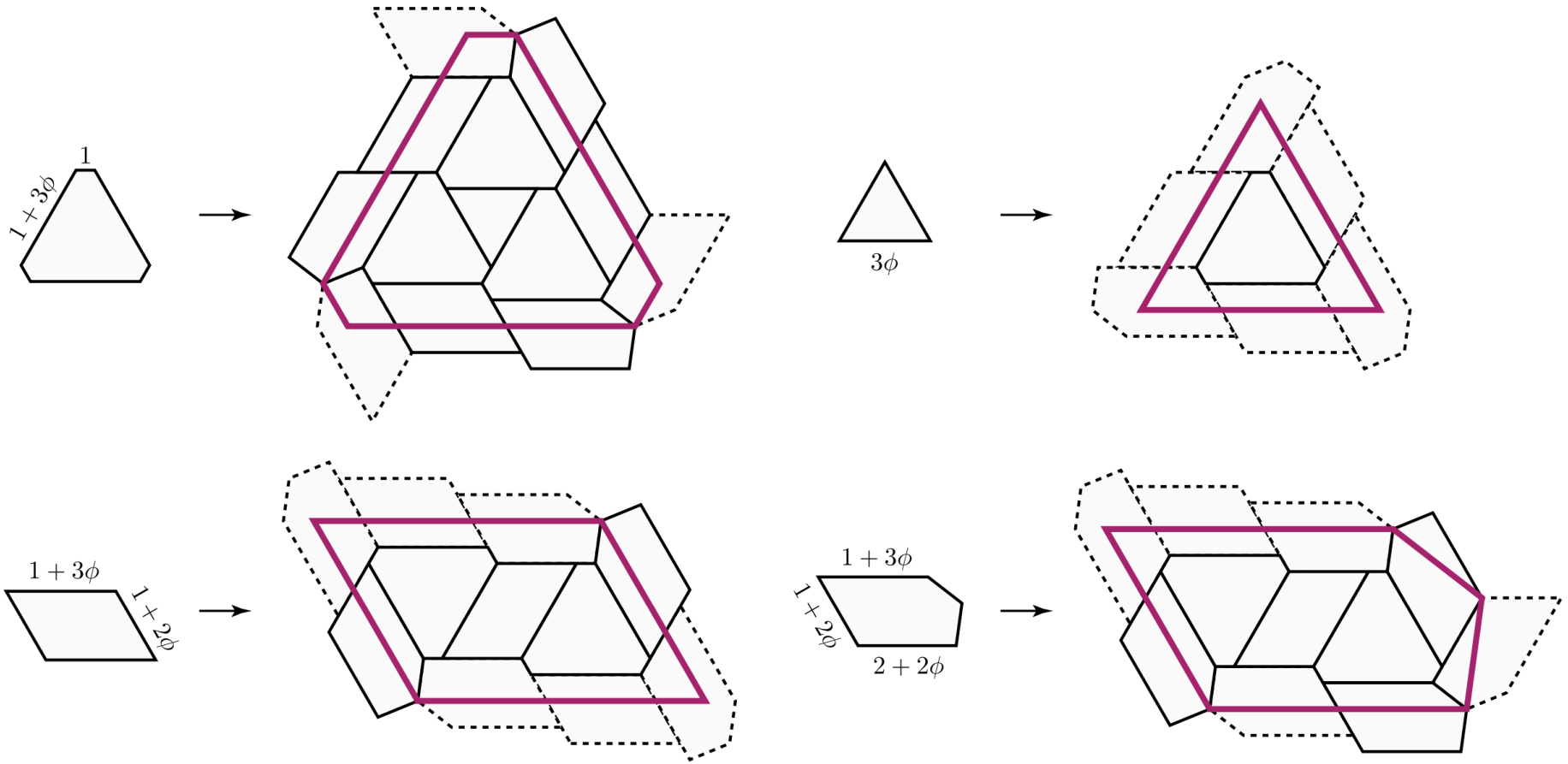
Figure 2.5: The H, T, P, and F metatiles (top), constructed by simplifying the boundaries of clusters of hats. We mark the H, T, and P metatiles with arrows when needed (bottom), to distinguish between otherwise symmetric orientations.

Metatiles in a tiling

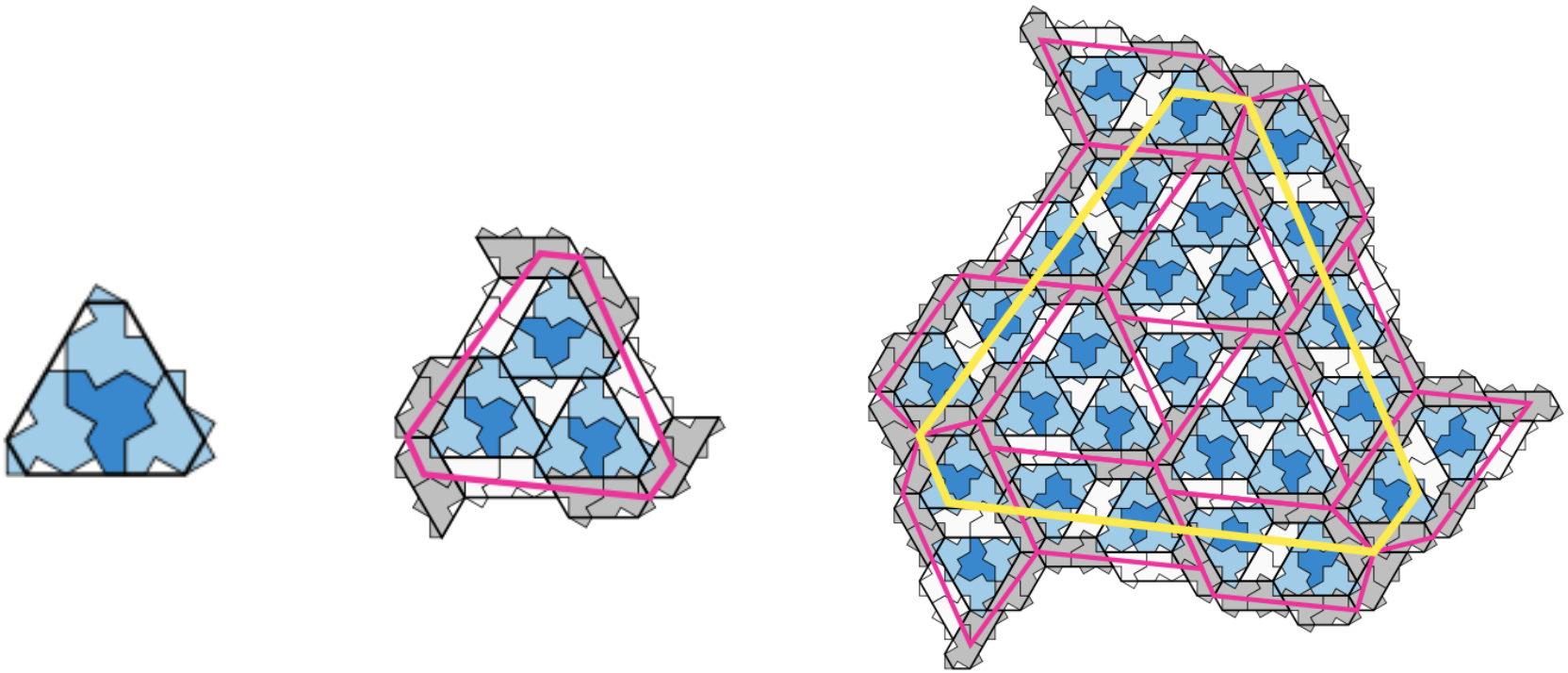




Expansion



Extension



Theorem (Tiling Extension)

If a shape can tile any size disk, then it can tile the whole plane.

Want more?

- `https://eschermath.org`
  Math & The Art of M. C. Escher. Anneke Bart, Bryan Clair (2012)
- The Tiling Book. Colin Adams (2022)
- Tilings and Patterns. Branko Grunbaum, G.C. Shephard (1987)