log(2)

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March 27, 2015

1666

COPERNICUS.
TYCHO BRAHE.
PTOLEMÆUS.
Progymnasmata. Tom. I.
Eccent.

Division

Division is a drag.

Example

$$\frac{192}{277}$$

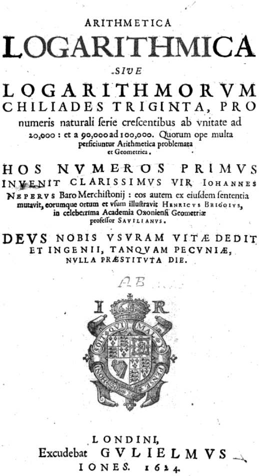
Multiples of 277:

1	277
2	554
3	831
4	1108
5	1385
6	1662
7	1939
8	2216
9	2493

Logarithms


MIRIFICI
Logarithmorum
Canonis descriptio,
Ejusque usus, in utraque Trigonometria; ut etiam in omni Logistica Mathematica, Amplissimi, Facillimi, & expeditissimi explicatio.
Authore ac Inventore, IOANNE NEPERO, Barone Merchistonii, &c. Scoto.
EDINBURGI, Ex officina ANDREÆ HART Bibliopolæ, cIɔ. DC. XIV.

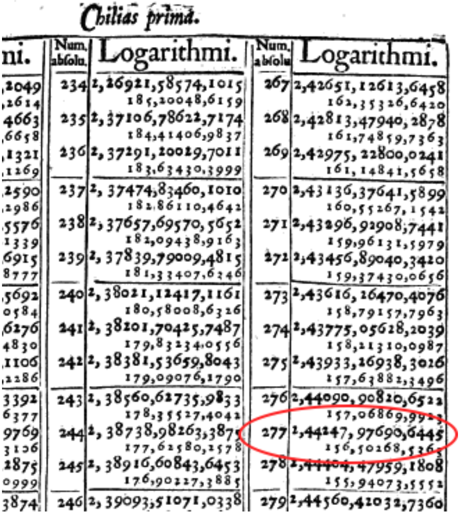
John Napier, 1614


ARITHMETICA
LOGARITHMICA
SIVE
LOGARITHMORVM CHILIADES TRIGINTA, PRO numeris naturali serie crescentibus ab vnitate ad 20,000: et a 90,000 ad 100,000. Quorum ope multa perficiuntur Arithmetica problemata et Geometrica.
HOS NVMEROS PRIMVS INVENIT CLARISSIMVS VIR IOHANNES NEPERVS Baro Merchistonij: eos autem ex eiusdem sententia mutavit, eorumque ortum et vsum illustravit HENRICVS BRIGGIVS, in celeberrima Academia Oxoniensi Geometriæ professor SAVILIANVS.
DEVS NOBIS VSVRAM VITÆ DEDIT ET INGENII, TANQVAM PECVNIÆ, NVLLA PRÆSTITVTA DIE.
LONDINI, Excudebat GVLIELMVS IONES. 1624.

Henry Briggs, 1624

Division with logarithms

$\log_{10} 277 \approx 2.442480; \quad \log_{10} 192 \approx 2.283301$



Chilias prima.

ni.	Num. absolu.	Logarithmi.	Num. absolu	Logarithmi.
,2049	234	2,36921,58574,1015	267	2,42651,12613,6458
,2614		185,20048,6159		162,35326,6420
4663	235	2,37106,78622,7174	268	2,42813,47940,2878
,6658		184,41406,9837		161,74859,7363
,1321	236	2,37291,20029,7011	269	2,42975,22800,0241
,1269		183,63430,3999		161,14841,5658
,2590	237	2,37474,83460,1010	270	2,43136,37641,5899
,2986		182,86110,4642		160,55267,1542
5576	238	2,37657,69570,5652	271	2,43296,92908,7441
1339		182,09438,9163		159,96131,5979
6915	239	2,37839,79009,4815	272	2,43456,89040,3420
8777		181,33407,6246		159,37430,0656
,5692	240	2,38021,12417,1161	273	2,43616,26470,4076
0584		180,58008,6326		158,79157,7963
,6276	241	2,38201,70425,7487	274	2,43775,05628,2039
4830		179,83234,0556		158,21310,0987
,1106	242	2,38381,53659,8043	275	2,43933,26938,3026
,2286		179,09076,1790		157,63882,3496
3392	243	2,38560,62735,9833	276	2,44090,90820,6522
6377		178,35527,4042		157,06869,99[illegible]
9769	244	2,38738,98263,3875	277	2,44247,97690,6445
3126		177,62580,2578		156,50268,5265
2875	245	2,38916,60843,6453	278	2,44404,47959,1808
0999		176,90227,3885		155,94073,5552
3874	246	2,39093,51071,0338	279	2,44560,42032,7360

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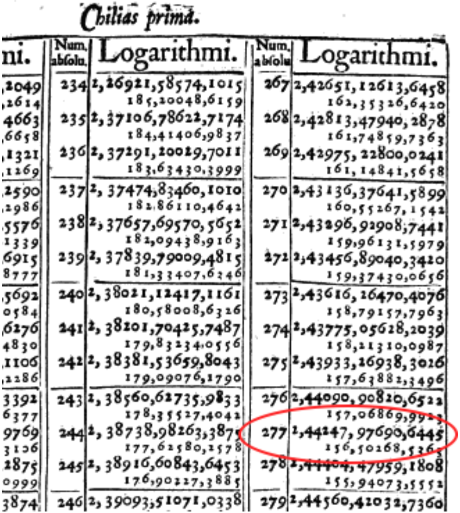

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$$\log_{10} \frac{1920}{277} \approx 3.28330 - 2.44247$$
$$= 0.84083$$

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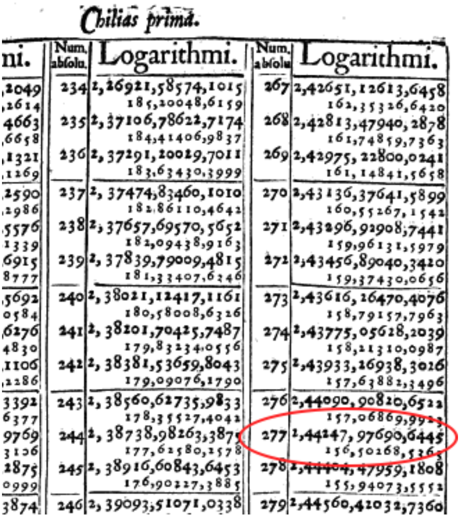

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$$\frac{192}{277} \approx 0.6931$$

Logarithmic Tables

- Build new logs from old:

$$\log(1920) = \log(2^7 \cdot 3 \cdot 5) = 7\log(2) + \log(3) + \log(5)$$

- Interpolate known values.

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Question

How to compute log(2)?
(by hand!)

Briggs, 1624

Knows how to compute square roots.
Observes that

$$\log_{10}(1+x) \approx Mx$$

when x is small. Then, compute:

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \log_{10} \sqrt{\sqrt{\cdots \sqrt{10}}}$$

After 54 steps, Briggs' $M = 0.43429448190325180{\color{red}4} \approx 1/\ln(10)$.

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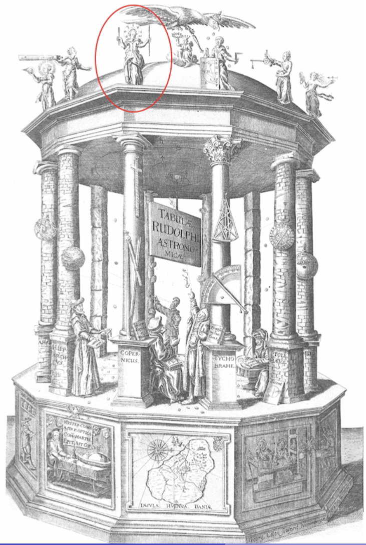
Briggs did 47 root extractions, computing $\log_{10}(2)$ to 19 decimal places.

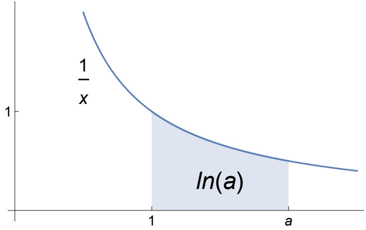
Johannes Kepler, 1627

The *Tabulae Rudolphinae Astronomica* (1627) contains natural (base e) logarithms, where $\ln(1+x) \approx x$ for small x.



"Hyperbolic" Logarithms

Late 1650's. Logarithms are related to hyperbolic areas.



Brouncker, 1668:

$$\ln(2) = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \frac{1}{9\cdot 10} + \cdots$$

These five terms give $\ln(2) \approx 0.645\ldots$

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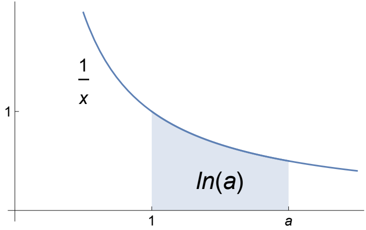

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These five terms give $\ln(2) \approx 0.645\ldots$.
It takes 80 terms for the next digit.

Nicolas Mercator, 1668

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

so

$$\begin{aligned}\ln(2) = \ln(1+1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\ &= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots\end{aligned}$$

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On the other hand

$$\ln(1.1) = .1 - .01\cdot\frac{1}{2} + .001\cdot\frac{1}{3} - .0001\cdot\frac{1}{4} + \cdots$$

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$\ln(1.1) = 0.09531\ 01798\ 04324\ 86004\ 39521\ 23280\ 76509\ 22206\ 054$

Newton, 1671, Method of fluxions

Use $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$ to compute

- $\ln(\frac{9}{10})$, $x = -0.1$
- $\ln(\frac{12}{10})$, $x = 0.2$
- $\ln(\frac{8}{10})$, $x = -0.2$

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$$\ln(2) = \ln\left(\frac{12 \cdot 12}{8 \cdot 9}\right) = 2\ln(\frac{12}{10}) - \ln(\frac{8}{10}) - \ln(\frac{9}{10})$$

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Can get $\ln(2) \approx 0.69314$ using only six terms of each series.
Newton computed 16 digits, needing 21 terms of each series.

Simpson's Rule

$$\ln(2) = \int_1^2 \frac{1}{x} dx$$

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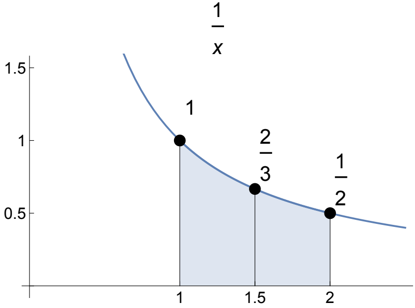
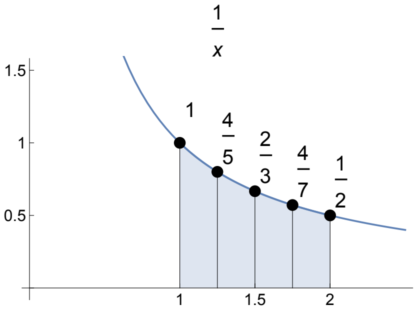
$$\int_1^2 \frac{dx}{x} \approx \frac{1}{6}\left(1 + 4\cdot\frac{2}{3} + \frac{1}{2}\right) = \frac{25}{36} = 0.694444\ldots = S_1$$



Simpson's Rule II

$$\int_1^2 \frac{dx}{x} \approx \frac{1}{2}\cdot\frac{1}{6}\left(1 + 4\cdot\frac{4}{5} + \frac{2}{3}\right) + \frac{1}{2}\cdot\frac{1}{6}\left(\frac{2}{3} + 4\cdot\frac{4}{7} + \frac{1}{2}\right) = \frac{1747}{2520} = 0.6932539\ldots = S_2$$



Extrapolation

The error in composite Simpson's rule is proportional to h^4, where h is the width of the subdivisions. Reducing h by a factor of 2 should reduce the error by a factor of 16.

$$\varepsilon_1 = S_1 - \log 2; \quad \varepsilon_2 = S_2 - \log 2$$

we expect $\varepsilon_2 \approx \frac{1}{16}\varepsilon_1$. Then

$$S_2 - \log 2 \approx \frac{1}{16}\left(S_1 - \log 2\right)$$

$$\frac{1}{15}\left(16 S_2 - S_1\right) \approx \log 2$$

$$\log 2 \approx 0.6931746\ldots$$

The L function

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 \cdots$$
$$-\ln(1-x) = x + x^2/2 + x^3/3 + x^4/4 + x^5/5 \cdots$$

So that

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + x^3/3 + x^5/5 + \cdots$$

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Put

$$L(k) = \frac{1}{2}\ln\left(\frac{1+1/k}{1-1/k}\right) = \frac{1}{2}\ln\left(\frac{k+1}{k-1}\right)$$

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$$L(k) = \frac{1}{k} + \frac{1}{3k^3} + \frac{1}{5k^5} + \frac{1}{7k^7} + \cdots$$

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Notice $2L(3) = \ln(2)$. Five terms of the series give $\ln(2) \approx 0.693144$.

Euler, 1748

$$L(k) = \frac{1}{k} + \frac{1}{3k^3} + \frac{1}{5k^5} + \frac{1}{7k^7} + \cdots$$

Euler observed

$$2L(5) + 2L(7) = \ln(\frac{6}{4}) + \ln(\frac{8}{6}) = \ln(\frac{8}{4}) = \ln(2)$$

and used this to compute ln(2) to 25 digits, which takes only 17 terms of each L series.

Records

- Napier, ~ 1615
- Briggs, 1624
- 16 digits, Newton, 1671
- 25 digits, Euler, 1748
- 48 digits, Wolfram, 1778 (48 digits for all numbers up to 2200)
- 137 digits, Shanks, 1853, $\ln(2) = 14L(31) + 10L(49) + 6L(161)$
- 260 digits, Adams, 1878, $\ln(2) = 7\ln(\frac{10}{9}) - 2\ln(\frac{25}{24}) + 3\ln(\frac{81}{80})$
- 330 digits, Uhler, 1940, linear combinations of $L(k)$.

Currently: 31 billion digits, Chan & Yee, 2009:

$$\ln(2) = 18L(26) - 2L(4801) + 8L(8749)$$

References

- Herman Goldstine. *A History of Numerical Analysis*, 1977, Springer-Verlag.
- Xavier Gourdon and Pascal Sebah, *The Logarithmic Constant:* $\log 2$, `numbers.computation.free.fr/Constants/constants.html`, 2010
- Denis Roegel, *A reconstruction of the tables of Briggs' Arithmetica logarithmica*, 2010.