Propositional Calculus

Here is a definition of the formal system for propositional logic.

1. Symbols:

   $A, B, C, D, \ldots, Z$(and optionally, allow primes, A', A'', etc.)

   $\sim, \vee, \wedge, (,)$

   and additionally the symbols $\Rightarrow$ and $\Leftrightarrow$, which are only shorthand for their equivalent forms (defined below).

2. Well Formed Formulas

   - Any letter $A, \ldots, Z$ is well formed (adding primes is also ok).
   - If x is WFF, then so is $\sim x$.
   - if x and y are WFF, then so is $(x \vee y)$.
   - if x and y are WFF, then so is $(x \wedge y)$.

3. Rules of Inference

   - Separation
     (a) From $(x \wedge y)$, form x.
     (b) From $(x \wedge y)$, form y.
   - Conjunction: From x and y, form $x \wedge y$.
   - Double Negation: x and $\sim\sim x$ are equivalent.
   - DeMorgan's Rules
     (a) $\sim (x \vee y)$ is equivalent to $(\sim x \wedge \sim y)$.
     (b) $\sim (x \wedge y)$ is equivalent to $(\sim x \vee \sim y)$.
   - Modus Ponens: From x and $x \Rightarrow y$, form y.
   - Deduction: Given a sequence of rules of inference that lead from x to y, form $x \Rightarrow y$.
   - Definition of $\Rightarrow$: $x \Rightarrow y$ is equivalent to $(\sim x \vee y)$.
   - Definition of $\Leftrightarrow$: $x \Leftrightarrow y$ is equivalent to $(x \Rightarrow y) \wedge (y \Rightarrow x)$.

4. Axioms: This system has no axioms.

Some Tautologies

Using Deduction, the rules of inference give rise to tautologies:

- Separation (a): $(P \wedge Q) \Rightarrow P$.
- Separation (b): $(P \wedge Q) \Rightarrow Q$.
- Conjunction: $(P \Rightarrow (Q \Rightarrow (P \wedge Q)))$.
- Double Negation: $P \Leftrightarrow \sim\sim P$.
- DeMorgan (a): $\sim (P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q)$.
- DeMorgan (b): $\sim (P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$.

In fact, a different approach to defining the formal system is to have only one Rule of Inference, Modus Ponens, and a bunch of axioms corresponding to these tautologies

Here are some fundamental tautologies:

- Addition: $P \Rightarrow (P \vee Q)$.
- Commutativity of $\wedge$: $(P \wedge Q) \Leftrightarrow (Q \wedge P)$.
- Commutativity of $\vee$: $(P \vee Q) \Leftrightarrow (Q \vee P)$.
- Commutativity of $\Leftrightarrow$: $(P \Leftrightarrow Q) \Leftrightarrow (Q \Leftrightarrow P)$.
- Associativity of $\wedge$: $((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$.
- Associativity of $\vee$: $((P \vee Q) \vee R) \Leftrightarrow (P \vee (Q \vee R))$.
- Distributivity 1: $(P \wedge (Q \vee R)) \Leftrightarrow ((P \wedge Q) \vee (P \wedge R))$.
- Distributivity 2: $(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$.
- Material Biconditional: $((P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\sim P \wedge \sim Q)))$

And a few more, which are more subtle:

- Transposition: $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$.
- Modus Tollens: $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$.
- Hypothetical Syllogism: $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$.
- Disjunctive Syllogism: $((P \vee Q) \wedge \sim Q) \Rightarrow P$.