Math 266 - Final Exam Practice Questions

1. Let p be a prime and $a \in \mathbb{Z}$. Here is a logical statement:
   "If p divides a^2, then p divides a."
   What is the contrapositive statement?

2. Let P and Q be propositions. Find a statement logically equivalent to $P \vee Q$ using only negation ($\sim$) and conjuction ($\wedge$) operations.

3. Prove that there is a rational number between any two unequal real numbers.

4. Define, for $x, y \in \mathbb{R}$:
   $$\max(x,y) = \begin{cases} x & \text{if } x \geq y; \\ y & \text{if } y > x. \end{cases}$$
   Prove, for any $z \in \mathbb{R}$, that if $z \geq x$ and $z \geq y$, then $z \geq \max(x,y)$.

5. For $x, y \in \mathbb{R}$, give a precise definition of $\min(x,y)$, the minimum of x and y. Prove that $\min(x,y) = -\max(-x,-y)$.

6. Let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two families of sets. Prove
   $$\left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{n=1}^{\infty} B_n\right) \subseteq \bigcap_{n=1}^{\infty} A_n \cup B_n.$$

7. Let R be a relation on a set A, and define a new relation S on A by xSy iff xRy or yRx. Prove that S is symmetric.

8. Define a relation on $\mathbb{Z}_7$ (the integers modulo 7) by xRy if $x \equiv y+1 \pmod 7$ or $x \equiv y+2 \pmod 7$. Draw the digraph for R.

9. For $a, m \in \mathbb{N}$, define $f : \mathbb{Z}_m \to \mathbb{Z}_m$ by $f(x) = ax$. Prove f is a bijection if and only if $\text{GCD}(m,a) = 1$.

10. Let $f : \mathbb{Q} \cap (0, \infty) \to \mathbb{N} \times \mathbb{N}$ by $f(p/q) = (p,q)$, where we assume p/q is a rational number in lowest terms.
    (a) Is f one-to-one? Prove or disprove.
    (b) Is f onto? Prove or disprove.

11. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(x)$.
    (a) Find $f([-14.000001, 182.632])$.
    (b) Find $f^{-1}([-14.000001, 182.632])$.

12. A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
    (a) Give three examples of even functions.
    (b) Prove: For any function $g : \mathbb{R} \to \mathbb{R}$, the function $f(x) = g(x^2)$ is even.

(c) Prove: If $f : \mathbb{R} \to \mathbb{R}$ is even, then there is a function $g : \mathbb{R} \to \mathbb{R}$ with $f(x) = g(x^2)$.

13. A class $a \in \mathbb{Z}_m$ is called a *quadratic residue* modulo m if there is $b \in \mathbb{Z}$ with $b^2 \equiv a \pmod{m}$. Find the quadratic residues modulo 13.

14. Find integers a, b, and c so that:
    - $c \not\equiv 0 \pmod{35}$
    - $ac \equiv bc \pmod{35}$
    - $a \not\equiv b \pmod{35}$.

15. Prove that if $a \equiv b \pmod{m}$ and $n|m$, then $a \equiv b \pmod{n}$.

16. Prove that $2^{44} - 1$ is divisible by 89.

17. Suppose a is not divisible by 17. Prove that either $a^8 + 1$ or $a^8 - 1$ is divisible by 17.

18. Define the numbers c_n by $c_1 = 1$, $c_2 = 1$, and for $n \geq 1$, $c_{n+2} = 1/(c_n + c_{n+1})$. Prove that $1/2 \leq c_n \leq 1$ for all $n \in \mathbb{N}$.

19. Let $f(x) = (1-x)^{-1}$. Prove by induction that the n^{th} derivative of f is given by $f^{(n)}(x) = n!(1-x)^{-n-1}$.

20. Give an example of two functions $f, g : \mathbb{N} \to \mathbb{N}$ so that f is not onto, but $g \circ f$ is onto.