ZETA FUNCTIONS OF GRAPHS WITH $\mathbb{Z}$ ACTIONS

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Abstract. Suppose Y is a regular covering of a graph X with covering transformation group $\pi = \mathbb{Z}$. This paper gives an explicit formula for the L^2 zeta function of Y and computes examples. When $\pi = \mathbb{Z}$, the L^2 zeta function is an algebraic function. As a consequence it extends to a meromorphic function on a Riemann surface. The meromorphic extension provides a setting to generalize known properties of zeta functions of regular graphs, such as the location of singularities and the functional equation.

1. Introduction

Given a finite graph, there is a zeta function which encodes some of the combinatorics of the graph. The zeta function was defined by Ihara and extended by Hashimoto and then Bass. Stark and Terras [6] give a fine introduction to the subject taking a geometric approach.

There is an analagous zeta function for any infinite graph with cofinite action of a discrete group. Let $Y = (VY, EY)$ be a locally finite (but typically infinite) graph and suppose the group π acts freely on Y with finite quotient graph X. Let P denote the set of free homotopy classes of primitive closed paths in Y. For $\gamma \in P$, $\ell(\gamma)$ is the length of the shortest representative of γ. The group π_γ is the stabilizer of γ under the action of π. The L^2 *zeta function* of Y is the infinite product

$$Z_Y^{(2)}(u)^{-1} = \prod_{\gamma \in \pi \backslash P} \left(1 - u^{\ell(\gamma)}\right)^{\frac{1}{|\pi_\gamma|}} . \tag{1.1}$$

This definition was first given in [2] as a specialization from a more general setting, but beware that the notation Z in [2] refers to the reciprocal of the zeta function considered here and elsewhere in the literature. See [5] for a more direct treatment of the case considered here.

Date: July 22, 2006.

For finite graphs, the fundamental theorem is the Ihara-Hashimoto-Bass rationality formula, which says that the zeta function is the reciprocal of a polynomial. The analogous theorem for infinte graphs requires techniques of von Neumann algebras. The infinte graph result is formally similar, and implies convergence of (1.1), but the L^2 zeta function is not typically a rational function.

Let δ be the adjacency operator of Y acting on $l^2(VY)$. For $f \in l^2(VY)$ let $Qf(v) = q(v)f(v)$ where $q(v)+1$ is the degree of the vertex v. Put $\Delta_u = I - \delta u + Qu^2$. Then from [2, Theorem 0.3],

$$Z_Y^{(2)}(u)^{-1} = (1-u^2)^{-\chi(X)} \operatorname{Det}_\pi \Delta_u \tag{1.2}$$

where Det_π is a von Neumann determinant defined in [2]. In particular, the product in (1.1) converges for small u (which was not a priori obvious).

In this paper, the only group considered is $\pi = \mathbb{Z}$, so that $X = Y/\mathbb{Z}$. Theorem 2.2 computes $Z_Y^{(2)}(u)$ in this case. The main difficulties to overcome are the evaluation of a particular definite integral and careful bookeeping with branches of multi-valued complex functions.

The formula for $Z_Y^{(2)}(u)$ is algebraic, and Theorem 2.3 takes advantage of this to extend $Z_Y^{(2)}(u)$ to a meromorphic function $\tilde{Z}$ defined on a compact Riemann surface S (which depends on Y). From another viewpoint, $Z_Y^{(2)}(u)$ is naturally a multi-valued meromorphic function defined on all of $\mathbb{C}$.

The surface S covers the Riemann sphere $\mathbb{C}P^1$ with branch points, and the branch points play a similar role for infinite graphs as the poles do for zeta functions of finite graphs. Specifically, Theorem 3.2 gives conditions for $\tilde{Z}$ of a $q+1$ regular graph Y to have all its branch points over the set

$$C = \{u \in \mathbb{C} : |u| = q^{-1/2}\} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1].$$

C is exactly the set where poles may occur for zeta functions of finite $q+1$ regular graphs.

The extension to $\tilde{Z}$ gives a meaningful context for functional equations relating $u \leftrightarrow \frac{1}{qu}$, and Section 3.1 explores these.

Finally, Section 4 gives a number of computations for specific Y.

This paper is intended as a model for how one might attack more general $\pi \neq \mathbb{Z}$. It is shown in [3] that for a q-regular graph, the L^2 zeta function always extends holomorphically to the interior of the set C. In the most optimistic scenario, the L^2 zeta function is always algebraic and therefore extends past C to a compact Riemann surface. More likely, one may need to allow noncompact surfaces with infinitely many sheets over $\mathbb{C}P^1$. In the worst scenario, the "branch points" could spread out continuously over C and prevent any further extension of domain. In any event, the explicit computation that provides the key here is not likely to unlock the more general case.

The author would like to thank David Moulton and Shahriar Mokhtari for helpful discussions.

1.1. **Group Von Neumann Algebras.** For completeness, here is a quick overview of relevant material from von Neumann algebras. For π a countable discrete group, the *von Neumann algebra* of π is the algebra $\mathcal{N}(\pi)$ of bounded π-equivariant operators from $l^2(\pi)$ to $l^2(\pi)$.

The *von Neumann trace* of an element $f \in \mathcal{N}(\pi)$ is defined by

$$\mathrm{Tr}_\pi f = \langle f(e), e \rangle$$

for $e \in \pi$ the unit element. The group ring $\mathbb{C}[\pi]$ is contained in $\mathcal{N}(\pi)$, acting on $l^2(\pi)$ by right multiplication. It is a dense subspace. The trace of an element of the group ring is simply the coefficient of the identity.

For $H = \oplus_{i=1}^n l^2(\pi)$ and a bounded π-equivariant operator $f : H \to H$, define

$$\mathrm{Tr}_\pi f = \sum_{i=1}^n \mathrm{Tr}_\pi f_{ii}.$$

The trace as defined is independent of the decomposition of H. The determinant $\mathrm{Det}_\pi \Delta(Y, u)$ is defined via formal power series as $(\mathrm{Exp} \circ \mathrm{Tr}_\pi \circ \mathrm{Log})\Delta(Y, u)$ and converges for small u.

Example 1.1. When $\pi = \mathbb{Z}$, Fourier transform identifies $l^2(\mathbb{Z})$ with $L^2(S^1)$. An element $\sum_{n=-\infty}^{\infty} c_n t^n \in \mathcal{N}(\pi)$ tranforms to multiplcation by $f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, and

$$\mathrm{Tr}_\pi f = \langle f \cdot 1, 1 \rangle = \int_{S^1} f(\theta) d\theta = c_0.$$

2. Graphs with $\mathbb{Z}$ actions

We assume $\pi = \mathbb{Z} = \langle t \rangle$, the free abelian group on one generator t. Suppose $X = Y/\mathbb{Z}$ has v vertices. Choosing lifts of these vertices to Y, we identify

$$l^2(VY) = \bigoplus_v l^2(Z)$$

and the adjacency operator δ is then a $v \times v$ matrix with entries in the group ring $\mathbb{Z}[\langle t \rangle]$. Since δ is self-adjoint, it satisfies $\delta(t) = \delta(t^{-1})^T$. Similarly, $\Delta_u(t) = \Delta_u(t^{-1})^T$ (but beware that Δ_u is not generally self-adjoint). Therefore, $\operatorname{Det} \Delta_u \in \mathbb{C}[\langle t \rangle]$ is symmetric in t and t^{-1}, and we can write $\operatorname{Det} \Delta_u = P_u(\frac{t+t^{-1}}{2})$ for some polynomial P_u. The coefficients of P_u are integer polynomials in u.

We know in general that $\operatorname{Det}_\pi \Delta_u$ is idependent of the choice of lifts of vertices, but here it is very clear, since choosing a different lift will multiply a row by t^k and the corresponding column by t^{-k} (for some k). In particular, P_u depends only on Y and the $\mathbb{Z}$ action.

Now, under Fourier transform, $\bigoplus_v l^2(Z) = \bigoplus_v L^2(S^1)$. Here $S^1 = \{e^{i\theta} | \theta \in (-\pi, \pi]\}$ with measure normalized to have total measure 1. Under Fourier tranform, multiplication by t becomes multiplication by the function $e^{i\theta}$, and hence Δ_u is represented by a $v \times v$ matrix which will be denoted $M_u(\theta)$. To compute the zeta function,

$$\operatorname{Det}_\pi \Delta_u = \exp \operatorname{Tr}_\pi \operatorname{Log}(\Delta_u) \tag{2.1}$$

$$= \exp \int_{S^1} \operatorname{Tr} \operatorname{Log}(M_u(\theta)) d\theta \tag{2.2}$$

$$= \exp \int_{S^1} \log \det(M_u(\theta)) d\theta \tag{2.3}$$

$$= \exp \int_{S^1} \log P_u(\cos(\theta)) d\theta. \tag{2.4}$$

2.1. **The Line.** To proceed further, we work out a crucial example. Let $VY = \mathbb{Z}$, and connnect n to $n+1$ with an edge (so Y is a line). Then

$$\Delta_u = 1 - (t + t^{-1})u + u^2,$$

$$M_u(\theta) = 1 - 2\cos(\theta)u + u^2,$$

and

$$P_u(x) = 1 - 2ux + u^2.$$

Notice that for $|u| < 1$ and for all θ, $M_u(\theta) \notin (-\infty, 0]$. In what follows, log will be the principal branch of the logarithm.

Now restrict to $|u| < 1$. Because Y has no loops, the L^2 zeta function for Y is identically 1. Therefore, by [2, Theorem 0.3]

$$\begin{aligned} 1 &= (1-u^2)^0 \operatorname{Det}_\pi \Delta_u \\ &= \exp \int_{S^1} \log(1 - 2u\cos(\theta) + u^2) d\theta \qquad (2.5) \\ &= \exp \int_{S^1} \log(2u) + \log(\frac{u+u^{-1}}{2} - \cos(\theta)) d\theta \qquad (2.6) \\ &= 2u \exp \int_{S^1} \log(r - \cos(\theta)) d\theta. \qquad (2.7) \end{aligned}$$

Here, we have assumed $u \neq 0$ and put $r = (u + u^{-1})/2$. Generally, some care must be taken when writing $\log(xy) = \log(x) + \log(y)$. If $u \in (-1, 0)$, then $r - \cos(\theta) < 0$ and the identity is off by $2\pi i$. However, the $2\pi i$ is washed out by the exp in front. For other values of u there is no problem, because the imaginary parts of u and r have opposite sign.

Notice that $r = \cosh(-\log(u))$, so that $u = e^{-\operatorname{arccosh}(r)}$. Here, arccosh has a branch cut discontinuity on $(-\infty, 1]$ and range $\{a + bi | a > 0, b \in (-\pi, \pi]\} \cup [0, \pi]i$.

Proposition 2.1. *Suppose $r \in \mathbb{C}$. Then*

$$\int_{S^1} \log(r - \cos(\theta)) d\theta = \operatorname{arccosh}(r) - \log(2). \qquad (2.8)$$

Proof. The discussion above proves that for $r \in \mathbb{C} - [-1, 1]$,

$$\exp \int_{S^1} \log(r - \cos(\theta)) d\theta = \frac{1}{2} e^{\operatorname{arccosh}(r)}.$$

Taking the log of both sides,

$$\int_{S^1} \log(r - \cos(\theta)) d\theta = \operatorname{arccosh}(r) - \log(2) \qquad (2.9)$$

In principle, this is only true up to $2\pi i k$ for some $k \in \mathbb{Z}$. However, k must be zero since both log and arccosh have imaginary part in the range $(-\pi, \pi]$.

Now we extend (2.9) to all of $\mathbb{C}$. We check the imaginary part explicitly. For $r \in [-1, 1]$, put $r = \cos(\phi)$, $\phi \in [0, \pi]$. Then $\Im(\operatorname{arccosh}(r) - \log 2) = \phi$. On the other hand, $\arg(r-\cos(\theta))=\pi$ when $\cos(\theta)>r$ and is 0 otherwise. Therefore,

$$\int_{S^1}\Im(\log(r-\cos(\theta)))d\theta=\int_{S^1}\arg(r-\cos(\theta))d\theta \tag{2.10}$$

$$=\pi\cdot m\{\theta|\cos(\theta)>\cos(\phi)\} \tag{2.11}$$

$$=\pi\cdot\frac{2\phi}{2\pi}=\phi. \tag{2.12}$$

Next, consider the real part of (2.9). The real part of the left hand side is

$$\int_{S^1}\log|r-\cos(\theta)|d\theta.$$

It is not hard to see that this integral is finite even for $r\in[-1,1]$. On the other hand, $\Re(\operatorname{arccosh}(r)-\log 2)$ is a continuous function on all of $\mathbb{C}$ (and equals $-\log 2$ on $[-1,1]$). Thus the two sides are defined on all of $\mathbb{C}$, equal on $\mathbb{C}-[-1,1]$, and the right side is continuous.

Now, for $r\in[-1,1], n=1,2,\ldots$ put

$$f_n(\theta)=\log|r-\cos(\theta)+i/n|.$$

The f_n are a decreasing sequence of functions, bounded above (by $\sqrt{5}$), and converging a.e. to $\log|r-\cos(\theta)|$. By the Monotone Convergence Theorem,

$$-\log 2=\lim_{n\to\infty}\Re(\operatorname{arccosh}(r+i/n)-\log 2) \tag{2.13}$$

$$=\lim_{n\to\infty}\int_{S^1}f_n(\theta)d\theta \tag{2.14}$$

$$=\int_{S^1}\log|r-\cos(\theta)|d\theta, \tag{2.15}$$

□

Remark 1. Computing the integral in (2.8) is a good, difficult calculus exercise for $r>1$. I know of no elementary way to compute it in general.

Remark 2. The inverse hyperbolic cosine function satisfies

$$\operatorname{arccosh}(r)=\log\left(r+\sqrt{r+1}\sqrt{r-1}\right)$$

where the principal branches of arccosh, log, and $\sqrt{z}$ are used. In particular, $\Re(\operatorname{arccosh}(r))=\log\left|r+\sqrt{r+1}\sqrt{r-1}\right|$ is a continuous function. Taking the real part of both sides of (2.8) gives the integral

$$\int_{S^1}\log|r-\cos(\theta)|d\theta=\log\frac{1}{2}\left|r+\sqrt{r+1}\sqrt{r-1}\right|. \tag{2.16}$$

for all $r \in \mathbb{C}$.

2.2. **The explicit formula.**

Theorem 2.2. *Let Y be a regular $\mathbb{Z}$ covering of a finite graph X. Let $P_u(x)$ be the degree n polynomial so that*

$$\det \Delta_u = P_u\left(\frac{t+t^{-1}}{2}\right).$$

There is $R > 0$ so that for all $0 < |u| < R$,

$$Z_Y^{(2)}(u)^{-1} = (1-u^2)^{-\chi(X)}\frac{\alpha(u)}{2^n}\prod_i\left(r_i + \sqrt{r_i+1}\sqrt{r_i-1}\right). \tag{2.17}$$

Here $P_u(x) = \alpha(u)\prod_{i=1}^n(r_i(u)-x)$, and $r_i(u)$ are the roots of P_u. The square roots are principal, in the sense that $\sqrt{z} = \exp(\frac{1}{2}\log(z))$.

Proof. The polynomial $(-1)^n\alpha(u)$ is the coeffieicent of the top degree term x^n of P_u. Since $P_0 = 1$, 0 is a root of α. There is $R_1 > 0$ with $\alpha(u) \neq 0$ on $0 < |u| < R_1$, and so one can write $P_u(x) = \alpha(u)\prod_{i=1}^n(r_i(u)-x)$.

From (1.2), one need only compute $\mathrm{Det}_\pi \Delta_u$. There is a subtle point involving the log of a product, but the heart of the argument is the computation below, which begins with (2.4), and uses Proposition 2.1:

$$\mathrm{Det}_\pi \Delta_u = \exp\int_{S^1}\log P_u(\cos(\theta))d\theta \tag{2.18}$$

$$= \exp\int_{S^1}\log\alpha(u)\prod_{i=1}^n(r_i(u)-\cos(\theta))d\theta \tag{2.19}$$

$$= \exp\left(\log\alpha(u) + \sum_{i=1}^n\int_{S^1}\log(r_i(u)-\cos(\theta))d\theta\right) \tag{2.20}$$

$$= \exp\left(\log\alpha(u) + \sum_{i=1}^n(\operatorname{arccosh}(r_i)-\log(2))\right) \tag{2.21}$$

$$= \frac{\alpha(u)}{2^n}\prod_i\exp(\operatorname{arccosh}(r_i)) \tag{2.22}$$

$$= \frac{\alpha(u)}{2^n}\prod_i\left(r_i+\sqrt{r_i+1}\sqrt{r_i-1}\right). \tag{2.23}$$

It remains to justify the transition from (2.19) to (2.20).

Write

$$\log P_u(x) = \log\alpha(u) + \sum_{i=1}^n\log(r_i(u)-x) + 2\pi i k(u,x). \tag{2.24}$$

The function $k(u,x)$ is always an integer. We will show that $k(u,x) = k(u)$ is independent of $x \in [-1,1]$ and therefore pulls through the integral in (2.19) to be eaten by the exp.

Since $\Delta_u = I - \delta u + Qu^2$, we can write $P_u(x) = 1 + uT_u(x)$ for some polynomial T. Then there is $R_2 > 0$ so that for $|u| < R_2$ and $x \in [-1,1]$ we have $\Re(P_u(x)) > 0$. Therefore, $\log(P_u(x))$ is a continuous function of $x \in [-1,1]$.

In addition, for $0 < |u| < R_2$, we see that $P_u(x)$ has no roots on $[-1,1]$, *i.e.* $r_i(u) \notin [-1,1]$. Therefore $\log(r_i(u) - x)$ is a continuous function of $x \in [-1,1]$ (since we're using the principal branch of the logarithm).

We have shown that all other terms in (2.24) are continuous functions of x, and therefore $k(u,x)$ is a continuous function of x on $[-1,1]$, hence constant in x.

Setting $R = \min\{R_1, R_2\}$ completes the proof. □

2.3. **The meromorphic extension.** From Theorem 2.2, it is apparent that $Z_Y^{(2)}(u)$ is an algebraic function of u. In this section, we make this more explicit and then explore the consequences.

Let $s_i = \sqrt{r_i+1}\sqrt{r_i-1}$, and for $I = (\iota_1, \dots, \iota_n) \in \{\pm 1\}^n = \mathbb{Z}_2^n$, put

$$W_I = \prod_{i=1}^{n} r_i + \iota_i s_i.$$

Note that $(r_i + s_i)(r_i - s_i) = 1$ so that $W_I^{-1} = W_{-I}$. Theorem 2.2 then says that

$$Z_Y^{(2)}(u) = (1-u^2)^{\chi(X)} \frac{2^n}{\alpha(u)} W_{-1,-1\dots,-1}. \tag{2.25}$$

Let

$$\Omega(T) = \prod_{I \in \mathbb{Z}_2^n} (T - W_I). \tag{2.26}$$

Then Ω is a polynomial in T of degree 2^n. It is invariant under the transformation $s_i \to -s_i$, hence it is even degree in each s_i. We can replace s_i^2 with $r_i^2 - 1$ so that Ω is a degree n polynomial in r_i, symmetric in the r_i. This means that Ω is in fact a polynomial in the elementary symmetric functions $\sigma_1, \dots, \sigma_n$ of the r_i, for example:

$$\Omega(T) = \begin{cases} T^2 - 2\,T\,\sigma_1 + 1 & \text{for } n = 1, \\ T^4 - 4\,T^3\,\sigma_2 + T^2\,\left(-2 + 4\,{\sigma_1}^2 - 8\,\sigma_2\right) - 4\,T\,\sigma_2 + 1 & \text{for } n = 2, \end{cases} \tag{2.27}$$

and when $n = 3$,

$$\begin{aligned}\Omega(T) = T^8 &- 8T^7\sigma_3 + T^6\left(4 - 8{\sigma_1}^2 + 16\sigma_2 + 16{\sigma_2}^2 - 32\sigma_1\sigma_3\right)\\ &- T^5\left(-40\sigma_3 + 32{\sigma_1}^2\sigma_3 - 64\sigma_2\sigma_3\right)\\ &+ T^4\left(6 - 16{\sigma_1}^2 + 16{\sigma_1}^4 + 32\sigma_2 - 64{\sigma_1}^2\sigma_2 + 32{\sigma_2}^2 + 64\sigma_1\sigma_3 + 64{\sigma_3}^2\right)\\ &- \cdots - 8T\sigma_3 + 1\end{aligned} \tag{2.28}$$

using the symmetry of coefficients to finish (roots of Ω occur in reciprocal pairs).

Since the r_i are the roots of P_u,

$$\sigma_i = (-1)^{n-i}\left(\text{the } n - i^{\text{th}} \text{ coefficient of } \frac{P_u}{\alpha(u)}\right).$$

Thus σ_i is a rational function of u, and so $\Omega \in \mathbb{C}(u)[T]$.

We have shown that W_I and therefore $Z_Y^{(2)}(u)$ are algebraic functions of u of degree less than or equal to 2^n.

Theorem 2.3. *Let Y be a regular $\mathbb{Z} = \pi$ covering of a finite graph X. Then $Z_Y^{(2)}(u)$ extends uniquely to a meromorphic function on a Riemann surface.*

More precisely, there exists a compact Riemann surface S, a (branched) covering map $\Pi : S \to \mathbb{C}P^1$, and a meromorphic function $\tilde{Z}$ on S. There is a point $z_0 \in \Pi^{-1}(0)$ and a neighborhood U of z_0 on which Π is biholomorphic such that $\tilde{Z}(z) = Z_Y^{(2)}(\Pi(z))$ for all $z \in U$.

The triple $(S, \Pi, \tilde{Z})$ is unique in the following sense: If $(S', \Pi', \tilde{Z}')$ has the corresponding properties, then there exists exactly one fiber preserving biholomorphic mapping $\tau : S \to S'$ such that $\tilde{Z} = \tilde{Z}' \circ \tau$.

Remark 3. The number of sheets of Π is less than or equal to 2^n, where n is the degree of the polynomial P_u defined earlier.

Proof. Define W_I and Ω as above. The difficult work is finished, as we showed already that W_I is algebraic. Since $W_{-1,-1,\ldots,-1}$ is holomorphic in a neighborhood of 0 and $\Omega(W_{-1,-1,\ldots,-1}) = 0$, there is a unique irreducible factor $\Phi \in \mathbb{C}(u)[T]$ with

$$\Phi(W_{-1,-1,\ldots,-1}) = 0. \tag{2.29}$$

in a neighborhood of 0.

The algebraic function defined by $\Phi(T)$ consists of S and Π as above, plus a meromorphic function f on S such that $(\Pi^*\Phi)(f) = 0$. It is unique in the sense of fiber preserving biholomorphic mappings as above (see [4, I.8] for details).

Since $W_{-1,-1,\ldots,-1}$ is holomorphic in a neighborhood of 0, there is a point $z_0 \in \Pi^{-1}(0)$ and a neighborhood U of z_0 on which Π is biholomorphic with $f(z) = W_{-1,-1,\ldots,-1}(\Pi(z))$ for $z \in U$.

For $z \in S$, let $u = \Pi(z)$ and put

$$\tilde{Z}(z) = (1-u^2)^{\chi(X)} \frac{2^n}{\alpha(u)} f(z). \tag{2.30}$$

to complete the proof. □

3. Regular graphs

In this section, assume that X is $q+1$ regular.

3.1. **Functional Equations.** The zeta function for finite regular graphs satisifes a number of functional equations under the transformation

$$\tau : u \to \frac{1}{qu}$$

(see [6]). The situation for L^2 zeta functions is somewhat less simple.

First notice that

$$\Delta_{1/qu} = I - \delta\frac{1}{qu} + q\frac{1}{(qu)^2} = \frac{1}{qu^2}\left(I - \delta u + qu^2\right) = \frac{1}{qu^2}\Delta_u.$$

Then the polynomial $P_{1/qu}(x)$ has the same roots $r_1, \ldots, r_n$ as $P_u(x)$. Since Ω and $W_{-1,-1,\ldots,-1}$ are symmetric functions of the r's, they are invariant under τ.

Suppose Ω is irreducible, so that the L^2 zeta function is defined on the Riemann surface S for Ω by (2.30). Then the transformation $u \to \frac{1}{qu}$ induces a biholomorphic involution $\tilde{\tau} : S \to S$ so that $f \circ \tilde{\tau} = f$. It is then easy to find functional equations for $\tilde{Z}$. For example:

Proposition 3.1. *Suppose X is $q+1$-regular and Ω is irreducible. For $z \in S$ put $u = \Pi(z)$. Then*

$$\left(\tilde{Z} \circ \tilde{\tau}\right)(z) = q^{2e-v} u^{2e} \left(\frac{1-u^2}{q^2u^2-1}\right)^{-\chi} \tilde{Z}(z). \tag{3.1}$$

Here v and e are the number of vertices and edges of X, and $\chi = \chi(X) = v - e$. (Compare [1, Cor 3.10]*)*

Proof. This is a straightforward calculation using $f \circ \tilde{\tau} = f$, equation (2.30), and

$$\alpha(\frac{1}{qu}) = \left(\frac{1}{qu^2}\right)^v \alpha(u).$$

□

If Ω is reducible, one gets a collection of disjoint Riemann surfaces $S_1, \ldots, S_k$ and the map $\tilde{\tau}$ may permute them. We are interested in $\tilde{Z}$ on a particular choice S, and so it will not satisfy a functional equation in any traditional sense. The line (example 4.1) is a good example of this.

3.2. **Location of branch points.** The zeta function for a finite, $q+1$ regular graph has all of its poles in the set

$$C = \{u \in \mathbb{C} : |u| = q^{-1/2}\} \cup [-1, -\frac{1}{q}] \cup [\frac{1}{q}, 1].$$

For the L^2 zeta function, we can make a slightly weaker statement for branch points.

Theorem 3.2. *Let Y be a regular $\mathbb{Z} = \pi$ covering of a finite graph X. Suppose that X is $q+1$ regular. Let Ω be the polynomial defined in* (2.26)*, and assume Ω is irreducible. If the field extension $\mathbb{C}(u)[T]/(\Omega(T)) : \mathbb{C}(u)$ is Galois, then the covering Π from Theorem 2.3 has all of its branch points over C.*

Proof. Let D_0 and D_∞ be the connected components of $\mathbb{C} - C$. The L^2 zeta function $Z_\pi(Y, u)$ extends holomorphically to D_0, so the neighborhood U from Theorem 2.3 must also extend to cover D_0 with no branch points. The field extension $\mathbb{C}(u)[T]/(\Phi(T)) : \mathbb{C}(u)$ is Galois if and only if the deck transformations of S over $\mathbb{C}P^1$ act transitively on the sheets of S ([4, pg. 57]). Then $\Pi^{-1}(D_0)$ is a union of copies of U and has no branch points. The involution $\tilde{\tau}$ from the functional equation biholomorphically interchanges $\Pi^{-1}D_0$ with $\Pi^{-1}D_\infty$, so that Π can only be branched on C. □

In example 4.3, we will see a graph for which the L^2 zeta function is branched over 0 and the deck transformations of S are not transitive.

The assumption that Ω is irreducible is less well motivated. As in example 4.4, the zeta function for a graph with reducible Ω will still satisfy a functional equation if $\tilde{\tau}$ perserves S.

The following argument gives hope for a close relationship between branch points of $\tilde{Z}$ for Y and zeros of $Z(X)$. To compute the zeta function $Z(X)$ of the quotient graph $X = Y/\mathbb{Z}$, one takes the determinant of $\Delta_X(u) = I - \delta_X u + Qu^2$, where δ_X is the adjacency operator on X. Poles of $Z(X)$ occur when $\det \Delta_X(u) = 0$. But δ_X is equal to δ on Y under $t \to 1$, and so poles of $Z(X)$ occur when $P_u(1) = 0$, or equivalently when some root $r_i(u) = 1$.

If $r_i(u) = 1$ then the terms $r_i \pm \sqrt{r_i + 1}\sqrt{r_i - 1}$ coincide. In other words, two roots of Ω coincide at any u where $Z(X)$ has a pole – a necessary condition for S to be branched over u.

Frequently, branch points of $\tilde{Z}$ do coincide with poles of $Z(X)$. However, examples in the next section show that both possible implications are false in general.

4. Examples

Example 4.1 (The Line)**.** Let Y be the line, as in Section 2.1. We saw earlier that

$$P_u(x) = 1 + u^2 - 2ux.$$

Then $\alpha(u) = 2u$ and $r(u) = \frac{1+u^2}{2u}$. From (2.27), we have

$$\Omega(T) = T^2 - T\frac{1+u^2}{u} + 1 = \frac{(T-u)(Tu-1)}{u}.$$

Here Ω is reducible. Some careful computation shows that

$$W_{-1}(u) = \frac{1+u^2}{2u} - \sqrt{\frac{1+u^2}{2u} + 1}\sqrt{\frac{1+u^2}{2u} - 1} \quad = \begin{cases} u & \text{if } |u| < 1 \\ 1/u & \text{if } |u| > 1 \end{cases} \tag{4.1}$$

so $\Phi(T) = T - u$, the Riemann surface S is $\mathbb{C}P^1$, $f(u) = u$, and the zeta function is $2f/\alpha = 1$.

Notice that the transformation $\tau : u \to \frac{1}{qu}$ (here $q = 1$) interchanges the two irreducible surfaces. On the other surface, the analog of $\tilde{Z}$ is u^2, and in fact the functional equation (3.1) becomes

$$u^2 = 1^{2\cdot 1 - 1} u^{2 \cdot 1} \cdot 1 \cdot 1.$$

Example 4.2 (Some degree 1 graphs)**.** Let Y be the first graph shown in Table 4 (all these graphs take the obvious $\mathbb{Z}$ action). Y is 4-regular, so $q = 3$. It's quotient graph X is a vertex with two loops,and $\chi(X) = -1$.

The adjacency matrix for Y is the 1×1 matrix $(t^{-1} + 2 + t)$. Then $P(x) = -2ux + 1 - 2u + 3u^2$ which has the one root shown in the table. From (2.27),

$$\Omega(T) = T^2 - u^{-1}(1 - 2u + 3u^2)T + 1$$

which is irreducible.

The associated Riemann surface S is a two sheeted branched cover of $\mathbb{C}P^1$. Possible branch points occur when the discriminant of Ω vanishes, which happens in this case at

$$u = 1, u = \frac{1}{3}, u = \frac{i}{\sqrt{3}}, u = \frac{-i}{\sqrt{3}}.$$

Here, all four are in fact branch points of multiplicity 2. The pattern of branch points is shown the table, and the set C is also indicated.

The Riemann-Hurwitz formula gives the genus of a branched covering of $\mathbb{C}P^1$ as $g = b/2 - d + 1$ with d the number of sheets and b the total branching order. For this graph the genus is 1 and S is a torus.

Other lines of Table 4 give the results of similar computations for different Y with $n = 1$. In all cases, $\Omega(T) = T^2 - 2rT + 1$ is irreducible, S is a two sheeted branch cover, and all branch points are mulitiplicity 2.

Graph #3 is an example in which poles of the zeta function for the quotient graph do not correspond to branch points of S. In this graph #3 of the table, $r(-\frac{1}{4} \pm \frac{i}{4}\sqrt{7}) = 1$, but these are not branch points of the L^2 zeta function.

Graphs #2,4, and 5 are bipartite and have vertical bilateral symmetry. Graphs #4 and 5 have different zeta functions because they have different α and different χ.

Graph #6 is non-regular. It's branch points are shown with circles of radius $1/\sqrt{2}$ and $1/\sqrt{3}$ for scale.

Example 4.3 (A regular graph with branch point off of C)**.** Consider the 4-regular graph Y with vertices $\mathbb{Z} \cup \mathbb{Z}$ as shown in Figure 4.3. The adjacency matrix of Y is

$$\delta = \begin{pmatrix} t + t^{-1} & 1 + t^{-1} \\ 1 + t & t + t^{-1} \end{pmatrix}.$$

$P_u(x)$ is degree 2, and the two roots of P_u are

$$r_{\pm}(u) = \frac{1}{4u}\left(2 + u + 6u^2 \pm \sqrt{u(4 + 9u + 12u^2)}\right).$$

#	Graph	q	$\chi(X)$	$r(u)$	Branchpoints	Genus
1		3	-1	$\frac{1-2\,u+3\,u^2}{2u}$		1
2		4	-3	$\frac{1-9\,u^2+16\,u^4}{8\,u^2}$		3
3		2	-2	$\frac{1-u+u^2-3\,u^3+2\,u^4-4\,u^5+8\,u^6}{4\,u^3}$		3
4		3	-2	$\frac{1+3\,u^2}{4\,u}$		1
5		3	-1	$\frac{1+3\,u^2}{4\,u}$		1
6		NR	-2	$\frac{1-u+u^2-3\,u^3+6\,u^4}{4\,u^2}$		3

TABLE 1. Degree 1 Graphs

FIGURE 1. The graph and branchpoints for Example 4.3.

From (2.27), Ω is the irreducible degree 4 polynomial

$$\Omega(T) = T^4 - \frac{1+4u^2+9u^4}{u^2}T^3 + \frac{2+4\,u+15\,u^2+12\,u^3+18\,u^4}{u^2}T^2 - \frac{1+4u^2+9u^4}{u^2}T + 1.$$

The Riemann surface S has four sheets covering $\mathbb{C}P^1$. Evaluating the discriminant of Ω, one has 10 points u where $Z_Y^{(2)}$ has duplicate values. Checking the local behavior near those 10 points and additionally near $u = 0$, $u = \infty$, one finds that S is unbranched at four of them. At

$$u \in \left\{1, \frac{1}{3}, \pm\frac{i}{\sqrt{3}}\right\},$$

sheets of S come together in two pairs of multiplicity two branch points. At

$$u = \left\{0, -\frac{9}{24} \pm \frac{i}{24}\sqrt{111}, \infty\right\},$$

one pair of sheets come together in a multiplicity two branch point and the other two sheets are unbranched. The pattern of branchpoints is shown in Figure 4.3, and the genus of S is 3.

The most interesting thing here is that the zeta function is branched over 0 and ∞, which are not in the set C. Of course, the sheet corresponding to the original unextended definition of $Z_Y^{(2)}$ is not one of the two sheets that come together at $u = 0$. The group of deck transformations of S is not transitive, and the field extension $\mathbb{C}(u)[T]/(\Omega(T)) : \mathbb{C}(u)$ is not Galois.

FIGURE 2. The graph and branchpoints for Example 4.4.

Example 4.4 (A nontrivial reducible graph)**.** Let Y have vertices $\mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z}$ connected as shown in Figure 4.4. It is the graph Cartesian product of the line with a triangle.

Here Ω factors into a fourth degree term and the square of a quadratic. The factor Φ corresponding to $Z_Y^{(2)}$ is the fourth degree term, so S is four sheeted. There are twelve branch points:

$$u \in \left\{ \frac{1}{3}, 1, \frac{\pm i}{\sqrt{3}}, \frac{-3 \pm i\sqrt{3}}{6}, \frac{\pm 1 \pm i\sqrt{11}}{6}, -\frac{1}{4} + \frac{\sqrt{\frac{7}{3}}}{4} \pm \frac{i}{2}\sqrt{\frac{1}{2} + \frac{\sqrt{\frac{7}{3}}}{2}} \right\},$$

shown in Figure 4.4. At each u, sheets come together in two pairs of multiplicity two branch points, so the genus of S is 9.

Even though Ω is reducible, all branch points still lie on the set C. Here $Z_Y^{(2)}$ must still satisfy the functional equation (3.1), because the involution $\tilde{\tau}$ preserves S - the other two irreducible factors of Ω are degree 2 while Φ is degree 4.

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