RESIDUAL AMENABILITY AND THE APPROXIMATION OF L^2-INVARIANTS

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Abstract. We generalize Lück's Theorem to show that the L^2-Betti numbers of a residually amenable covering space are the limit of the L^2-Betti numbers of a sequence of amenable covering spaces. We show that any residually amenable covering space of a finite simplicial complex is of determinant class, and that the L^2-torsion is a homotopy invariant for such spaces. We give examples of residually amenable groups, including the Baumslag-Solitar groups.

Introduction

In 1994, Wolfgang Lück [15] proved the beautiful theorem that if X is a finite simplicial complex with residually finite fundamental group, the L^2-Betti numbers of the universal covering of X can be approximated by the ordinary Betti numbers of a sequence of finite coverings of X. In fact, the question of approximation dates back to Kazhdan [12] (see also [10, Pg. 231]) but only an inequality was known. Dodziuk and Mathai [8] have shown a result analogous to Lück's Theorem in the situation where the covering transformation group is amenable. Specifically, they show that the L^2-Betti numbers of an amenable covering $\widetilde{X}$ of X can be approximated by the ordinary Betti numbers of a sequence of Følner subsets of $\widetilde{X}$. This paper generalizes Lück's Theorem to the case where the cover of X has residually amenable transformation group, a large class of groups that includes the residually finite groups of Lück's Theorem and the amenable groups of Dodziuk and Mathai.

In this paper, we also consider L^2-torsion. At first, L^2-torsion was defined for L^2-acyclic covering spaces. The L^2-analytic torsion was first studied in [18] and [14], and L^2-Reidemeister-Franz torsion was first studied in [6], see also [16]. Equality of the combinatorial and analytic L^2-torsions was proven in 1996 [4].

In order to define these L^2-torsions, one needs to establish decay near zero of the spectral density function for the L^2-Laplacian. In the case of a residually finite covering, Lück [15] derives an elegant estimate on the spectral density functions for the finite covers, which in the limit gives the necessary decay for the combinatorial L^2-Laplacian. Lück also proves the homotopy invariance of L^2-combinatorial torsion in this case.

In [5], the combinatorial and analytic torsion invariants were defined more generally as volume forms on L^2-cohomology, the decay condition on the spectrum now replaced by a similar condition known as determinant class. The results of [4] extend to show the equality of these more general combinatorial and analytic L^2-torsions.

Date: January 17,1999.

Dodziuk and Mathai [8] show that coverings with amenable covering group are of determinant class. Mathai and Rothenberg [19] have recently extended Lück's results to prove the homotopy invariance of L^2-torsion in that case. Although an error in their proof has been pointed out, the error is easily fixable and is corrected in the Appendix of this paper.

A main result of this paper is that coverings with residually amenable covering group are of determinant class, and that L^2-torsion of such spaces is a homotopy invariant.

On a different note, Farber [9] has generalized Lück's Theorem in a new direction, viewing it as a statement about flat bundles rather than finite coverings. In particular, he gives precise conditions for the convergence of L^2-Betti numbers of finite non-regular covers. A reasonable direction for future work would be to try and extend the results of this paper using his techniques.

We now formulate the main results of this paper. Let Y be a connected simplicial complex. Suppose that a finitely generated group π acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex.

Suppose there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. Form $Y_n = Y/\Gamma_n$, so that $Y_1, Y_2, \ldots$ are a tower of covering spaces of X.

Say that π is residually finite if there exist Γ_n's as above, so that the quotients π/Γ_n are all finite. Then each Y_n is a finite complex, and Lück's Theorem [15] states that

$$b_j^{(2)}(Y : \pi) = \lim_{n\to\infty} \frac{1}{|\Gamma_n|} b_j(Y_n)$$

where $b_j^{(2)}(Y : \pi)$ is the jth L^2-Betti number of Y, and $b_j(Y_n)$ is the ordinary jth Betti number of Y_n.

We generalize Lück's Theorem to the situation where π is residually amenable, meaning there exist Γ_n's as above so that the quotients π/Γ_n are all amenable. The first main result of this paper is

Theorem 0.1 (Approximation Theorem)**.** *Suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. If π is residually amenable, then*

$$b_j^{(2)}(Y : \pi) = \lim_{n\to\infty} b_j^{(2)}(Y_n : \pi/\Gamma_n).$$

The next result gives more evidence for the determinant class conjecture, which states that any regular covering space of a finite simplicial complex is of determinant class. For π residually finite this follows from [15], and it was shown for π amenable in [8].

Theorem 0.2 (Determinant Class Theorem)**.** *Suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. If π is residually amenable, then Y is of determinant class.*

Now we turn to the problem of homotopy invariance of L^2-torsion. Let M and N be compact cell complexes, $\widetilde{M}$ and $\widetilde{N}$ regular π-covering spaces. As in [19], a homotopy equivalence $f : M \to N$ induces a canonical isomorphism $\widetilde{f}_*^* : \det \overline{H}_{(2)}^*(\widetilde{N}) \to \det \overline{H}_{(2)}^*(\widetilde{M})$ of determinant lines of L^2-cohomology.

Let $\phi_{\widetilde{M}} \in \det \overline{H}_{(2)}^*(\widetilde{M})$ denote the combinatorial L^2-torsion of M.

Theorem 0.3 (Homotopy Invariance of L^2-Torsion)**.** *Suppose $f : M \to N$ is a homotopy equivalence of compact cell complexes, and $\widetilde{M}$ and $\widetilde{N}$ are regular covering spaces with residually amenable covering group. Then via the above identification of determinant lines of L^2-cohomology,*

$$\phi_{\widetilde{M}} = \phi_{\widetilde{N}} \in \det \overline{H}^*_{(2)}(\widetilde{M}).$$

This provides more evidence for the conjecture in [19] (see also [15]) that L^2-torsion is always a homotopy invariant when the covering spaces in question are of determinant class.

This paper is organized as follows. The first section covers preliminaries on residually amenable groups, and exhibits interesting examples. The second section proves the main technical theorem. It essentially states that the L^2-spectra of Laplacians on the Y_n approximate the L^2-spectrum of Laplacian on Y. Finally, the third section proves the three main theorems.

The results of this paper first appeared in the author's thesis [7]. T. Schick has independently arrived at similar results. Thanks to Mel Rothenberg for suggesting this direction of research, and also to Kevin Whyte and Shmuel Weinberger for helpful discussions.

1. Preliminaries

1.1. **Residual Properties.**

Definition 1.1. Let $\mathcal{C}$ be a nonempty class of groups (though possibly containing only one group). A group π is *residually* $\mathcal{C}$ if for any element $g \in \pi$, $g \neq e$, there exists a quotient group $\pi'(g)$ belonging to $\mathcal{C}$ such that $g \mapsto g' \in \pi'(g)$ with $g' \neq e$.

This paper will use a condition equivalent to residuality which holds for certain classes, for example any $\mathcal{C}$ which is closed under products and subgroups. For such a class, a countable group π is residually $\mathcal{C}$ if and only if there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that π/Γ_n belongs to $\mathcal{C}$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. For other basic theorems concerning residual properties of groups, we refer to [17].

When $\mathcal{C} = \{\text{finite groups}\}$ we say that π is a *residually finite* group.

1.2. **Amenability.**

Let π be a finitely generated discrete group, with word metric d. We use the following characterization of amenability, due to Følner.

Definition 1.2. π is *amenable* if there is a sequence of finite subsets $\{\Lambda_k\}_{k=1}^{\infty}$ such that for any fixed $\delta > 0$

$$\lim_{k\to\infty} \frac{\#\{\partial_\delta \Lambda_k\}}{\#\{\Lambda_k\}} = 0$$

where $\partial_\delta \Lambda_k = \{\gamma \in \pi : d(\gamma, \Lambda_k) < \delta \text{ and } d(\gamma, \pi - \Lambda_k) < \delta\}$ is a δ-neighborhood of the boundary of Λ_k.

Examples of amenable groups include finite groups, abelian groups, nilpotent groups and solvable groups, and groups of subexponential growth. Amenability for discrete groups is preserved by the following five processes [20, Prop. 0.16]:

1. Taking subgroups;
2. Forming quotient groups;
3. Forming group extensions by amenable groups;

4. Forming upward directed unions of amenable groups;
5. Forming a direct limit of amenable groups.

Free groups with two or more generators, and fundamental groups of closed negatively curved manifolds are *not* amenable.

1.3. **Residual Amenability.**

Definition 1.3. If π is residually $\mathcal{C}$, where $\mathcal{C} = \{\text{amenable groups}\}$, we say that π is *residually amenable*.

Recall that the derived subgroups $\pi^{(i)}$ of a group π are defined by $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. Say that π is *solvable* if $\pi^{(i)} = \{e\}$ for some i. The *rank* of π is then defined to be the smallest i for which $\pi^{(i)} = \{e\}$.

Contained in the class of residually amenable groups is the class of residually solvable groups. Residually solvable groups are naturally characterized in Prop 1.1. Free products of residually solvable groups are themselves residually solvable. This follows from the fact that solvability is a root property as discussed in [17].

Proposition 1.1. *π is residually solvable if and only if $\bigcap_{i=1}^{\infty} \pi^{(i)} = \{e\}$.*

Proof. Since $\pi/\pi^{(i)}$ is solvable, if $\bigcap_{i=1}^{\infty} \pi^{(i)} = \{e\}$ then π is residually solvable. Now suppose π is residually solvable, and let $g \in \bigcap_{i=1}^{\infty} \pi^{(i)}$. For any map $f : \pi \to S$ with S solvable of rank k, we have $f(g) \in f(\pi^{(i)}) \subseteq S^{(i)} = \{e\}$ for $i > k$. Since the image of g is trivial in any solvable quotient of π we must have $g = e$. □

Proposition 1.2. *If Γ is residually solvable, S is solvable, and π is an extension*

$$1 \to \Gamma \xrightarrow{\iota} \pi \xrightarrow{\kappa} S \to 1$$

then π is residually solvable.

Proof. Suppose S has rank k. Then $\kappa(\pi^{(k)}) \subseteq S^{(k)} = \{e\}$, so $\pi^{(k)} \subseteq \Gamma$ and therefore $\pi^{(k+i)} \subseteq \Gamma^{(i)}$ for all $i \geq 0$. Then $\bigcap_{i=1}^{\infty} \pi^{(i)} \subseteq \bigcap_{i=1}^{\infty} \Gamma^{(i)} = \{e\}$. □

Example 1.1. For nonzero integers p and q, define the *Baumslag-Solitar* group $BS(p,q)$ by

$$BS(p,q) = \left\langle a, b \mid a^{-1}b^p a = b^q \right\rangle.$$

A group π is *Hopfian* if $\pi/\Gamma \cong \pi$ implies $\Gamma = \{e\}$. The family of groups $BS(p,q)$ were first defined in [1], where it was shown that $BS(p,q)$ is Hopfian if and only if p and q are *meshed*, which means $p|q$, $q|p$, or p and q have exactly the same set of prime divisors. As any finitely generated residually finite group is Hopfian, the groups $BS(p,q)$ are not residually finite when p and q are not meshed.

Kropholler shows in [13] that the second derived subgroup $\pi^{(2)}$ is free when $\pi = BS(p,q)$, for any p, q. When p and q are not both ± 1, $\pi^{(2)}$ is free on two or more generators and therefore π is not amenable. However, π is residually solvable since it is an extension

$$1 \to \pi^{(2)} \to \pi \to \pi/\pi^{(2)} \to 1$$

with $\pi^{(2)}$ residually solvable and $\pi/\pi^{(2)}$ solvable (of rank 2).

More generally, let π be any non-cyclic group which is the fundamental group of a graph of infinite cyclic groups. Then from [13], $\pi^{(2)}$ is free and the above argument shows π is residually solvable.

Example 1.2. We show that the amalgamated free product of two abelian groups is residually solvable. Slightly more generally, suppose we have two solvable groups A and B, and two monomorphisms from an abelian group H into the centers of A and B given by $\alpha : H \to Z(A)$ and $\beta : H \to Z(B)$. Then the free product with amalgamation $A *_H B$ is residually solvable.

To see this, let $N = \{(\alpha(h), \beta(h)^{-1}) | h \in H\} \subset A \times B$. Since H is abelian, N is a subgroup of $A \times B$. N is normal because H includes into the centers of both A and B. Note that if H is only known to be normal in A and B, N is unlikely to be normal in $A \times B$.

Let K be the kernel of the natural map $A *_H B \to (A \times B)/N$. In $A *_H B$, K has trivial intersection with all conjugates of A and B, hence K is free by a well known theorem of group actions on trees (see for example [2, pg 54]). Then $A *_H B$ is an extension of the solvable group $(A \times B)/N$ by the residually solvable group K, and so $A *_H B$ is residually solvable.

Example 1.3. It is shown in [21] that any HNN-extension of a finitely generated abelian group is residually solvable.

Example 1.4. R.J. Thompson, G. Higman, K. Brown, and E.A. Scott have produced various classes of finitely presented infinite simple groups (see [22]). The example of Scott contains a free subgroup on two generators and is therefore not amenable and not residually amenable.

2. Main Technical Theorems

As in the introduction, suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. Suppose we have a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. Define $Y_n = Y/\Gamma_n$.

Suppose X has a_j cells in dimension j, and choose a lift to Y of each j-cell of X. These choices give a basis over $l^2(\pi)$ of the space $C^j_{(2)}(Y)$ of j dimensional l^2-cochains on Y. The lifts also descend to give bases of $C^j_{(2)}(Y_n)$ over $l^2(\pi/\Gamma_n)$.

Denote by Δ and Δ_n the Laplacian on $C^j_{(2)}(Y)$ and $C^j_{(2)}(Y_n)$ respectively. All arguments to follow will apply to a specific value of j, but this dependence will not be indicated.

Let $\{E(\lambda) : \lambda \in [0, \infty)\}$ and $\{E_n(\lambda) : \lambda \in [0, \infty)\}$ denote the (right continuous) family of spectral projections of Δ and Δ_n. Since Δ is π-equivariant, so are $E(\lambda) = \chi_{[0,\lambda]}(\Delta)$ for $\lambda \in [0, \infty)$. Similarly, $E_n(\lambda)$ are π/Γ_n-equivariant. Let $F, F_n : [0, \infty) \to [0, \infty)$ denote the spectral density functions

$$F(\lambda) = \mathrm{Tr}_\pi E(\lambda)$$

$$F_n(\lambda) = \mathrm{Tr}_{\pi/\Gamma_n} E_n(\lambda).$$

We now set

$$\begin{array}{rclrcl} \overline{F}(\lambda) & = & \limsup_{n\to\infty} F_n(\lambda); & \underline{F}(\lambda) & = & \liminf_{n\to\infty} F_n(\lambda) \\ \overline{F}^+(\lambda) & = & \lim_{\delta\to 0^+} \overline{F}(\lambda+\delta); & \underline{F}^+(\lambda) & = & \lim_{\delta\to 0^+} \underline{F}(\lambda+\delta). \end{array}$$

With the above notation, we state the main technical results of this paper.

Theorem 2.1. *For all $\lambda \in [0, \infty)$,*

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda)$$

Theorem 2.2. *Suppose there is some right continuous function $s : [0, \varepsilon) \to [0, \infty)$, with $s(0) = 0$ and so that for all n and for all $\lambda \in [0, \varepsilon)$ we have*

$$F_n(\lambda) - F_n(0) \le s(\lambda)$$

then

1. *$\overline{F}(\lambda)$ and $\underline{F}(\lambda)$ are right continuous at zero and one has the equalities*

$$\overline{F}(0) = \overline{F}^+(0) = F(0) = \underline{F}(0) = \underline{F}^+(0).$$

2. *For all $\lambda \in [0, \varepsilon)$,*

$$F(\lambda) - F(0) \le s(\lambda).$$

These theorems and their proofs are similar to Lück [15, Theorem 2.3], but here they are stated so as to require no conditions on the quotient groups π/Γ_n. The residual finiteness condition in Lück's Theorem or the residual amenability condition of this paper are required to provide s, the uniform decay at zero of the spectral density functions for the covers Y_n.

To show the two technical theorems, we first prove a number of preliminary lemmas.

Lemma 2.1. *There exists a number $K > 1$ such that the operator norms of Δ and Δ_n are smaller than K for all n.*

Proof. Choosing lifts of cells of X, we have identified the space of l^2-cochains on Y with $\bigoplus_{i=1}^a l^2(\pi)$. The combinatorial Laplacian Δ is then described by an $a \times a$ matrix B with entries in $\mathbb{Z}[\pi]$, acting by right multiplication. The Laplacian Δ_n is described by the same matrix B, now acting by right multiplication on $\bigoplus_{i=1}^a l^2(\pi/\Gamma_n)$

For $u = \sum_{g\in\pi} \lambda_g \cdot g \in \mathbb{C}[\pi]$ define $|u|_1 = \sum_{g\in\pi} |\lambda_g|$. Choose $K > 1$ so that

$$K \ge a \cdot \sum_{j=1}^{b} \max\{|B_{ij}|_1; i = 1, \dots, a\}.$$

The proof then proceeds exactly as in [15, Lemma 2.5]. □

Lemma 2.2. *Let $p(\mu)$ be a polynomial. There is a number n_0, depending only on the system of groups $\pi = \Gamma_1 \supset \Gamma_2 \supset \dots$ such that for all $n \ge n_0$*

$$\operatorname{Tr}_\pi p(\Delta) = \operatorname{Tr}_{\pi/\Gamma_n} p(\Delta_n)$$

Proof. We identify Δ with an $a \times a$ matrix B with entries in $\mathbb{Z}[\pi]$, as in the previous Lemma.

Fix elements $g_0, g_1, \dots, g_r \in \pi$ and $\lambda_0, \lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that $g_0 = e$, $g_i \ne e$, and $\lambda_i \ne 0$ for $1 \le i \le r$ so that

$$\sum_{j=1}^{a} (p(B))_{j,j} = \sum_{i=0}^{r} \lambda_i g_i.$$

Then

$$\operatorname{Tr}_\pi p(\Delta) = \lambda_0.$$

The Laplacian Δ_n on Y_n is also described by the matrix B, now acting on $\bigoplus_{i=1}^a l^2(\pi/\Gamma_n)$ by right multiplication. Then

$$\mathrm{Tr}_{\pi/\Gamma_n} p(\Delta_n) = \sum_{i=1}^r \lambda_i \, \mathrm{Tr}_{\pi/\Gamma_n} R(g_i)$$

where $R(g_i) : l^2(\pi/\Gamma_n) \to l^2(\pi/\Gamma_n)$ is right multiplication with g_i.

Since the intersection of the Γ_i's is trivial, there is a number n_0 such that for $n \geq n_0$ none of the elements g_i for $1 \leq i \leq r$ lies in Γ_n. Since Γ_n is normal, we conclude for $n \geq n_0$ and $i \neq 0$

$$\mathrm{Tr}_{\pi/\Gamma_n} R(g_i) = 0.$$

Then for $n \geq n_0$

$$\mathrm{Tr}_\pi \, p(\Delta) = \lambda(0) = \mathrm{Tr}_{\pi/\Gamma_n} p(\Delta_n).$$

□

Lemma 2.3. *Let K be as in Lemma 2.1. Let $\{p_k(\mu)\}_{k=1}^\infty$ be a sequence of polynomials, uniformly bounded on $[0, K]$, such that for the characteristic function $\chi_{[0,\lambda]}(\mu)$ of the interval $[0, \lambda]$,*

$$\lim_{k\to\infty} p_k(\mu) = \chi_{[0,\lambda]}(\mu)$$

holds for each $\mu \in [0, K]$. Then

$$\lim_{k\to\infty} \mathrm{Tr}_\pi \, p_k(\Delta) = F(\lambda).$$

Proof. This lemma and its proof are identical to [15, Lemma 2.7] □

We now prove Theorem 2.1. Fix $\lambda \geq 0$. Define for $k \geq 1$ a continuous function $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(\mu) = \begin{cases} 1 + \frac{1}{k} & \mu \leq \lambda \\ 1 + \frac{1}{k} - k(\mu - \lambda) & \lambda \leq \mu \leq \lambda + \frac{1}{k} \\ \frac{1}{k} & \lambda + \frac{1}{k} \leq \mu \end{cases}$$

Clearly $\chi_{[0,\lambda]}(\mu) < f_{k+1}(\mu) < f_k(\mu)$ and $f_k(\mu)$ converges to $\chi_{[0,\lambda]}(\mu)$ for all $\mu \in [0, \infty)$. For each k choose a polynomial p_k such that $\chi_{[0,\lambda]}(\mu) < p_k(\mu) < f_k(\mu)$ holds for all $\mu \in [0, K]$, where K is as in Lemma 2.1. The polynomials p_k satisfy the conditions of Lemma 2.3.

Because $\chi_{[0,\lambda]}(\mu) \leq p_k(\mu)$ for all $\mu \in [0, \|\Delta_n\|]$, we have

$$\begin{aligned} F_n(\lambda) &= \mathrm{Tr}_{\pi/\Gamma_n}\big(\chi_{[0,\lambda]}(\Delta_n)\big) \\ &\leq \mathrm{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big). \end{aligned} \tag{2.1}$$

On the other hand, we have $p_k(\mu) \le 1 + \frac{1}{k}$ for $\mu \in [0, \lambda + \frac{1}{k}]$ and $p_k(\mu) \le \frac{1}{k}$ for $\mu \in [\lambda + \frac{1}{k}, K]$. So

$$\begin{aligned}
\operatorname{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big) &\le \operatorname{Tr}_{\pi/\Gamma_n}\big((1+\tfrac{1}{k})\chi_{\left[0,\lambda+\frac{1}{k}\right]}(\Delta_n)\big) \\
&\quad + \operatorname{Tr}_{\pi/\Gamma_n}\big((\tfrac{1}{k})\chi_{\left[\lambda+\frac{1}{k},K\right]}(\Delta_n)\big) \\
&= (1+\tfrac{1}{k})F_n(\lambda+\tfrac{1}{k}) \\
&\quad + \tfrac{1}{k}(F_n(K) - F_n(\lambda+\tfrac{1}{k})) \\
&= F_n(\lambda+\tfrac{1}{k}) + \tfrac{1}{k}F_n(K)
\end{aligned} \tag{2.2}$$

Now notice $F_n(K) = \operatorname{Tr}_{\pi/\Gamma_n}\big(\chi_{[0,K]}(\Delta_n)\big)$. But $\chi_{[0,K]}(\Delta_n)$ is the identity on the space $C^j_{(2)}(Y_n)$, which is identified with $\bigoplus_{i=1}^{a_j} l^2(\pi/\Gamma_n)$. Thus

$$F_n(K) = a_j. \tag{2.3}$$

By Lemma 2.2, there is a number $n_0(k)$ for each polynomial p_k such that for $n \ge n_0(k)$

$$\operatorname{Tr}_\pi p_k(\Delta) = \operatorname{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big).$$

Then for $n \ge n_0(k)$, the equations (2.1), (2.2), and (2.3) give

$$F_n(\lambda) \le \operatorname{Tr}_\pi p_k(\Delta) \le F_n(\lambda + \tfrac{1}{k}) + \tfrac{1}{k}a_j$$

Taking limits as $n \to \infty$,

$$\overline{F}(\lambda) \le \operatorname{Tr}_\pi p_k(\Delta) \le \underline{F}(\lambda + \tfrac{1}{k}) + \tfrac{1}{k}a_j$$

Taking limits as $k \to \infty$, and using Lemma 2.3,

$$\overline{F}(\lambda) \le F(\lambda) \le \underline{F}^+(\lambda).$$

We have for all $\epsilon > 0$

$$F(\lambda) \le \underline{F}^+(\lambda) \le \underline{F}(\lambda+\epsilon) \le \overline{F}(\lambda+\epsilon) \le F(\lambda+\epsilon)$$

and since $\lim_{\epsilon\to 0+} F(\lambda+\epsilon) = F(\lambda)$ we get

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda).$$

This finishes the proof of Theorem 2.1. ☐

Next we show Theorem 2.2. We suppose there is some right continuous function $s : [0,\varepsilon) \to [0,\infty)$, with $s(0) = 0$ and so that for all n and for all $\lambda \in [0,\varepsilon)$ we have

$$F_n(\lambda) - F_n(0) \le s(\lambda).$$

Taking the limit inferior and limit superior for $n \to \infty$ gives:

$$\underline{F}(\lambda) \le \underline{F}(0) + s(\lambda) \text{ and } \overline{F}(\lambda) \le \overline{F}(0) + s(\lambda).$$

Taking the limit for $\lambda \to 0$ gives

$$\underline{F}^+(0) \le \underline{F}(0) \text{ and } \overline{F}^+(0) \le \overline{F}(0).$$

And finally, since $\underline{F}$ and $\overline{F}$ are increasing,

$$\underline{F}^+(0) = \underline{F}(0) \text{ and } \overline{F}^+(0) = \overline{F}(0).$$

We already know $\overline{F}^+(0) = F(0) = \underline{F}^+(0)$ from Theorem 2.1, and this proves Theorem 2.2.1. Since s is right continuous, we conclude:

$$\overline{F}^+(\lambda) \le F(0) + s(\lambda)$$

and Theorem 2.2.2 follows from Theorem 2.1. This finishes the proof of Theorem 2.2. ☐

The following pair of Lemmas are proved in [15, Lemma 3.3]. We will need them in the proof of Theorem 0.2 in the last section.

Lemma 2.4. *Suppose* $\lim_{n\to\infty} F_n(0) = F(0)$ *(which follows from Theorem 2.2 when the decay hypothesis is satisfied). Then*

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq \liminf_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

Lemma 2.5. *Suppose the* F_n *have a uniform spectral gap, that is there is some* $\epsilon > 0$ *so that* $F_n(\lambda) = F_n(0)$ *for all* n *and for all* $\lambda \leq \epsilon$*. Then*

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \geq \limsup_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

and therefore using the previous Lemma,

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda = \lim_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda.$$

Remark. In Lemma 2.5, Lück gives a sharper condition than simply demanding a spectral gap, but it will not be needed here.

3. Proofs Of The Main Theorems

In this chapter, we prove the approximation theorem, determinant class theorem, and homotopy invariance of L^2-torsion for residually amenable groups.

3.1. The Approximation Theorem.

Proof of Theorem 0.1 (Approximation Theorem).

Observe that the jth L^2-Betti numbers of Y and Y_n are given by

$$b^j_{(2)}(Y : \pi) = F(0); \quad b^j_{(2)}(Y_n : \pi/\Gamma_n) = F_n(0).$$

Therefore Theorem 0.1 will follow directly from Theorem 2.2.1, if we can establish a uniform decay of F_n near zero.

Since π/Γ_n is amenable, [8, Theorem 2.1.3] applies, and we have a constant $K > 1$ so that

$$F_n(\lambda) - F_n(0) \leq a_j \frac{\log K}{-\log \lambda} = s(\lambda) \tag{3.1}$$

for all $0 < \lambda < 1$. The constant K can be any number larger than $\max(\|\Delta_n\|, 1)$, and therefore can be chosen independently of n, by Lemma 2.1. ☐

Remark. In the case of Lück's Theorem, the groups π/Γ_n are finite, so the Laplacian on Y_n is a finite, self-adjoint matrix. Lück then proves, for any self-adjoint matrix, an elegant estimate on the number of eigenvalues which are less than a fixed λ. The estimate is weakened if the product of the nonzero eigenvalues is small, but this product must be at least one as the Laplacian has integer entries.

Dodziuk and Mathai make use of the same fundamental estimate when proving the approximation theorem for amenable groups.

3.2. **Results For Manifolds.** Suppose M is a compact Riemannian manifold, and $\widetilde{M}$ is a regular covering space for M with residually amenable transformation group π. Let $\widetilde{\Delta} = d^*d + dd^*$ denote the Laplacian on L^2 j-forms on $\widetilde{M}$. It is π-equivariant and essentially self-adjoint, so has a spectral family of π-equivariant projections $\{\widetilde{E}_j(\lambda) | \lambda \in [0,\infty)\}$. Each $E_j(\lambda)$ has Schwartz kernel, and one can then form the analytic spectral density function $\widetilde{F}_j(\lambda) = \mathrm{Tr}_\pi \widetilde{E}_j(\lambda)$.

Following arguments in [8], one can investigate the analytic spectral density function $\widetilde{F}(\lambda)$ of $\widetilde{\Delta}$. To relate this analytic Laplacian to the combinatorial situation in the previous sections, let X be a triangulation of M and lift to get a triangulation Y of $\widetilde{M}$.

The spectrum of $\widetilde{\Delta}$ is said to have a *gap at zero* if the spectral projection $\widetilde{E}(\lambda) = \widetilde{E}(0)$ for some $\lambda > 0$. Because the von Neumann dimension is faithful, $\widetilde{\Delta}$ has a spectral gap at zero if and only if $\widetilde{F}(\lambda) = \widetilde{F}(0)$ for some $\lambda > 0$.

Theorem 3.1. *(Gap Criterion) The spectrum of $\widetilde{\Delta}$ has a gap at zero if and only if there is a $\lambda > 0$ such that*

$$\lim_{n\to\infty} F_n(\lambda) - F_n(0) = 0.$$

Proof. Since $\widetilde{F}$ and F are dilation equivalent [11], $\widetilde{\Delta}$ will have a spectral gap if and only if Δ does. By Theorem 2.1, $F(\lambda) = F(0)$ if and only if

$$\lim_{n\to\infty} F_n(\lambda) - F_n(0) = 0.$$

□

Theorem 3.2. *(Spectral Density Estimate) There are constants $C > 0$ and $\varepsilon > 0$ independent of λ, such that for all $\lambda \in (0,\varepsilon)$*

$$\widetilde{F}(\lambda) - \widetilde{F}(0) \le \frac{C}{-\log\lambda}.$$

Proof. This follows directly from dilation equivalence and the estimate 3.1. □

The spectral density estimate is interesting as it provides evidence supporting the well known conjecture that the Novikov-Shubin invariants of a closed manifold are positive.

3.3. **The Determinant Class Theorem.** Recall that a covering space Y of a finite simplicial complex X is of determinant class if for each j,

$$-\infty < \int_{0^+}^{1} \log\lambda dF_j(\lambda),$$

where $F_j(\lambda)$ denotes the von Neumann spectral density function of the combinatorial Laplacian Δ_j on L^2 j-cochains.

We will prove that every residually amenable covering of a finite simplicial complex is of determinant class. The appendix of [3] contains a proof that every residually finite covering of a compact manifold is of determinant class. Their proof is based on Lück's approximation of von Neumann spectral density functions. Since an analogous approximation holds in the setting of this paper, we can apply the argument of [3] to prove Theorem 0.2. The fact that our coverings are infinite necessitates some modifications.

Proof of Theorem 0.2 (Determinant Class Theorem).

As with the rest of this paper, this proof will proceed for a fixed j which will be suppressed in the notation.

Denote by $\mathrm{Det}_{\pi/\Gamma_n} \Delta'_n$ the Fuglede-Kadison determinant of Δ_n restricted to the orthogonal complement of its kernel. It is given by the following Lebesgue-Stieltjes integral,

$$\log \mathrm{Det}_{\pi/\Gamma_n} \Delta'_n = \int_{0^+}^{\infty} \log \lambda dF_n(\lambda) = \int_{0^+}^{K} \log \lambda dF_n(\lambda)$$

with K as in Lemma 2.1. That is, $\|\Delta_n\| \leq K$.

As Y_n is an amenable cover of X, we are in the situation of Dodziuk-Mathai [8]. Their proof of the Determinant Class Theorem for amenable coverings [8, Thm 0.2] shows that

$$\log \mathrm{Det}_{\pi/\Gamma_n} \Delta'_n \geq 0,$$

which is stronger than simply determinant class - the key point being that the bound is uniform in n. It is worth remarking that their proof and the corresponding proof in [3] require a similar statement for the Laplacians on finite approximations. In the finite case, the Laplacian is an integer matrix, so the product of its nonzero eigenvalues is a positive integer and therefore at least 1.

By Lemma 3.1 at the end of this section,

$$0 \leq \log \mathrm{Det}_{\pi/\Gamma_n} \Delta'_n = (\log K)\big(F_n(K) - F_n(0)\big) - \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

which gives

$$\int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \leq (\log K)\big(F_n(K) - F_n(0)\big).$$

Now from Lemma 2.4,

$$\begin{aligned} \int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda &\leq \liminf_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \\ &\leq \liminf_{n\to\infty} (\log K)\big(F_n(K) - F_n(0)\big). \end{aligned} \tag{3.2}$$

Since we have uniform decay (3.1) of the F_n near 0, Theorem 2.2.1 applies and $\lim_{n\to\infty} F_n(0) = F(0)$. Since $K \geq \|\Delta_n\|$ for all n and $K \geq \|\Delta\|$,

$$F_n(K) = F(K) = a_j$$

for all n, so (3.2) becomes

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq (\log K)\big(F(K) - F(0)\big). \tag{3.3}$$

This shows in particular that the left hand integral exists, and again applying Lemma 3.1,

$$\log \mathrm{Det}_{\pi} \Delta' = (\log K)\big(F(K) - F(0)\big) - \int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \geq 0.$$

Since this is true for all j, Y is of determinant class. □

Lemma 3.1. *Let $F : [0, \infty) \to [0, \infty)$ be any non-decreasing function satisfying $\lim_{\lambda \to 0} F(\lambda) = 0$. Then for any $K > 0$,*

$$\int_{0^+}^{K} \log \lambda dF(\lambda) = (\log K)(F(K)) - \int_{0^+}^{K} \frac{F(\lambda)}{\lambda} d\lambda$$

where the left hand integral exists if and only if the right hand integral exists.

Proof. Fixing $1 > \epsilon > 0$, integration by parts yields

$$\int_{\epsilon}^{K} \log \lambda dF(\lambda) = (\log K)(F(K)) - (\log \epsilon)(F(\epsilon)) - \int_{\epsilon}^{K} \frac{F(\lambda)}{\lambda} d\lambda. \tag{3.4}$$

Now $-(\log \epsilon)(F(\epsilon))$ is non-negative, and $\int_{\epsilon}^{K} \log \lambda dF(\lambda)$ is decreasing as $\epsilon \to 0$. Thus, if $\int_{0^+}^{K} \frac{F(\lambda)}{\lambda} d\lambda$ exists then so does $\int_{0^+}^{K} \log \lambda dF(\lambda)$.

On the other hand, if $\int_{0^+}^{K} \log \lambda dF(\lambda)$ exists, then

$$0 \le -(\log \epsilon)(F(\epsilon)) = -(\log \epsilon) \int_{0^+}^{\epsilon} dF(\lambda) \le -\int_{0^+}^{\epsilon} \log \lambda dF(\lambda)$$

which goes to zero as $\epsilon \to 0$. This shows $\lim_{\epsilon \to 0} -(\log \epsilon)(F(\epsilon)) = 0$ and the Lemma follows. □

3.4. **Homotopy Invariance of L^2-Torsion.** The Fuglede-Kadison determinant Det_π induces a homomorphism from the Whitehead group of π,

$$\Phi_\pi : \mathrm{Wh}(\pi) \to \mathbb{R}^+$$

which was defined in [16] and [15]. Given a homotopy equivalence $f : M \to N$ of compact cell complexes, we choose f to be a cellular homotopy equivalence, and let M_f be the cellular mapping cone. Putting $\pi = \pi_1(M)$, the cochain complex $C^*(M_f)$ is an acyclic complex over the group ring $\mathbb{Z}[\pi]$, and defines the Whitehead torsion $T(f) \in \mathrm{Wh}(\pi)$. Then $\Phi_\pi(T(f)) \in \mathbb{R}^+$.

Now suppose π is residually amenable, so M and N are of determinant class. Let $\phi_{\widetilde{M}}$, $\phi_{\widetilde{N}}$ denote the L^2-torsion of M and N. The homotopy equivalence f canonically identifies the determinant lines of L^2-cohomology of $\widetilde{M}$ and $\widetilde{N}$, so that $\phi_{\widetilde{M}} \otimes \phi_{\widetilde{N}}^{-1} \in \mathbb{R}^+$.

Then from [19, Prop. 2.1], one has

$$\phi_{\widetilde{M}} \otimes \phi_{\widetilde{N}}^{-1} = \Phi_\pi(T(f)) \in \mathbb{R}^+.$$

Therefore, Theorem 0.3 is reduced to the following

Theorem 3.3. *Suppose that π is a finitely presented residually amenable group. Then the homomorphism*

$$\Phi_\pi : \mathrm{Wh}(\pi) \to \mathbb{R}^+$$

is trivial.

Proof. We can represent an arbitrary element of $\mathrm{Wh}(\pi)$ as the Whitehead torsion of a homotopy equivalence $f : L \to K$ of finite cell complexes, which without loss of generality is an inclusion. Let $\widetilde{L}$ and $\widetilde{K}$ denote the corresponding regular π covering complexes. The relative cochain complex $C^*(\widetilde{K}, \widetilde{L})$ is acyclic, and so is its L^2-completion $C^*_{(2)}(\widetilde{K},\widetilde{L})$. In particular, the combinatorial Laplacian $\Delta_j^{\widetilde{K},\widetilde{L}}$ is invertible and we see that

$$\Phi_\pi(T(f)) = \prod_{j=0}^{n} \mathrm{Det}_\pi(\Delta_j^{\widetilde{K},\widetilde{L}})^{\frac{(-1)^j j}{2}} > 0.$$

We claim that $\mathrm{Det}_\pi(\Delta_j^{\widetilde{K},\widetilde{L}}) = 1$ for each j. The j will be suppressed in the notation.

Form the covering spaces $\widetilde{L}_n$ and $\widetilde{K}_n$ with covering group π/Γ_n, and let $\Delta^{\widetilde{K}_n,\widetilde{L}_n}$ denote the Laplacian on $C^*_{(2)}(\widetilde{K}_n,\widetilde{L}_n)$.

As π/Γ_n is amenable, the work of Mathai-Rothenberg applies to the pair $(\widetilde{K}_n,\widetilde{L}_n)$. It follows from their proof [19, Prop. 2.6] that for all n,

$$\mathrm{Det}_{\pi/\Gamma_n} \Delta^{\widetilde{K}_n,\widetilde{L}_n} = 1.$$

Since $\Delta^{\widetilde{K},\widetilde{L}}$ is invertible we can choose a common bound K on $\|\Delta^{\widetilde{K},\widetilde{L}}\|$ and $\|(\Delta^{\widetilde{K},\widetilde{L}})^{-1}\|$, as in Lemma 2.1. K will also bound $\|\Delta^{\widetilde{K}_n,\widetilde{L}_n}\|$ and $\|(\Delta^{\widetilde{K}_n,\widetilde{L}_n})^{-1}\|$, and therefore and $\Delta^{\widetilde{K}_n,\widetilde{L}_n}$ has a spectral gap at zero of size at least K^{-1}.

Now by Lemma 2.5,

$$\mathrm{Det}_\pi \Delta^{\widetilde{K},\widetilde{L}} = \lim_{n\to\infty} \mathrm{Det}_{\pi/\Gamma_n} \Delta^{\widetilde{K}_n,\widetilde{L}_n} = 1.$$

□

This finishes the proof of Theorem 0.3. As remarked earlier, in both [15] and [19], the approximating spaces are finite. The approximating Laplacians have determinant 1 and spectral gap because they are finite invertible integer matrices, and the previous theorem essentially rests on this fact.

Appendix

Section 3.4 of this paper makes use of a similar argument in [19] which shows that L^2-torsion is a homotopy invariant of complexes with amenable fundamental group. The method in [19] contains a gap, and in this Appendix that gap will be corrected. The key idea for this correction is due to M. Rothenberg.

Consider complexes $X = Y/\pi$ as in this paper, but with π an amenable group. Choose a Følner exhaustion $\{\Lambda_n\}_{n=1}^\infty$ of π, and choose a subcomplex $\mathcal{F}$ of Y which is a fundamental domain for the action of π. Then Y_n is defined to be the finite subcomplex of Y consisting of all translates $g.\mathcal{F}$ for $g \in \Lambda_n$, and Δ_n is the Laplacian on the complex Y_n. Dodziuk and Mathai [8] show that renormalized spectra of Δ_n approximate the L^2-spectrum of Δ on Y.

Now define maps P_n and ι_n to be the orthogonal projection and the inclusion map between $C_{(2)}(Y) \supset C(Y_n)$. Set

$$\hat{\Delta}_n = P_n \circ \Delta \circ \iota_n : C(Y_n) \to C(Y_n).$$

As the first step towards fixing the problem in [19], notice that the approximation arguments in [8] work equally well with $\hat{\Delta}_n$. The two key points to notice are that $\hat{\Delta}_n$ is still a matrix with integer entries, and still agrees with Δ away from the boundary of Y_n.

Next consider the situation of [19], where X, Y, and Y_n are replaced by relative complexes (K,L), $(\widetilde{K},\widetilde{L})$ and $(\widetilde{K}_n,\widetilde{L}_n)$. We use the same notation Δ, Δ_n, and $\hat{\Delta}_n$

for the corresponding operators in this relative situation. Δ is invertible, and there are bounds

$$\|\Delta\| \leq K \text{ and } \|\Delta^{-1}\| \leq K.$$

The projection of Δ_n off of its kernel, Δ_n^+, is invertible. The error in [19] is the claim that there is a bound away from zero of the spectrum of Δ_n^+ which is uniform in n. In general this is not the case.

Here we show that there *is* a uniform bound away from zero for the spectra of $\hat{\Delta}_n$, $n = 1, 2, \ldots$. The rest of the argument in [19] is changed only by the replacement of Δ_n^+ with $\hat{\Delta}_n$. Notice that there is no longer any need to project off of the kernel as the operators $\hat{\Delta}_n$ are already invertible as shown below.

In what follows, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_n$ will denote the inner products on $C_{(2)}(\widetilde{K}, \widetilde{L})$ and $C(\widetilde{K}_n, \widetilde{L}_n)$. We have

$$\begin{aligned}
\inf \sigma(\hat{\Delta}_n) &= \inf_{c \in C(\widetilde{K}_n, \widetilde{L}_n); \|c\|=1} \left\langle \hat{\Delta}_n c, c \right\rangle_n \\
&= \inf_{c \in C(\widetilde{K}_n, \widetilde{L}_n); \|c\|=1} \langle P_n \Delta \iota_n c, c \rangle_n \\
&= \inf_{c \in C(\widetilde{K}_n, \widetilde{L}_n); \|c\|=1} \langle \Delta \iota_n c, \iota_n c \rangle \\
&\geq \inf_{c \in C_{(2)}(\widetilde{K}, \widetilde{L}); \|c\|=1} \langle \Delta c, c \rangle \\
&= \inf \sigma(\Delta) \\
&\geq K^{-1}.
\end{aligned}$$

Thus K^{-1} is a uniform lower bound on the spectra of $\hat{\Delta}_n$ as required to fix the proof in [19].

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