ZETA FUNCTIONS OF DISCRETE GROUPS ACTING ON TREES

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Abstract. This paper generalizes Bass' work on zeta functions for uniform tree lattices. Using the theory of von Neumann algebras, machinery is developed to define the zeta function of a discrete group of automorphisms of a bounded degree tree. The main theorems relate the zeta function to determinants of operators defined on edges or vertices of the tree.

A zeta function associated to a non-uniform tree lattice with appropriate Hilbert representation is defined. Zeta functions are defined for infinite graphs with a cocompact or finite covolume group action.

Introduction

The zeta function associated to a finite graph originated in work of Ihara [11, 10], which proves a structure theorem for torsion-free discrete cocompact subgroups of $PGL(2, \Bbbk_p)$ (where $\Bbbk_p$ is a p-adic number field or a field of power series over a finite constant field). If $\Gamma < PGL(2, \Bbbk_p)$ is such a group, Ihara shows that Γ is in fact a free group and defines a zeta function associated to Γ. An element $1 \neq \gamma \in \Gamma$ is called *primitive* if it generates its centralizer in Γ. Let $\mathcal{P}(\Gamma)$ denote the set of conjugacy classes of primitive elements of Γ. If λ_1 and λ_2 are the eigenvalues of a representative of $\gamma \in PGL(2, \Bbbk_p)$, define $l(\gamma) = |v_p(\lambda_1 \lambda_2^{-1})|$, where v_p is the normalized valuation for $\Bbbk_p$. Ihara defines the zeta function attached to Γ as:

$$Z_\Gamma(u) = \prod_{\gamma \in \mathcal{P}(\Gamma)} (1 - u^{l(\gamma)})^{-1}.$$

In [11], Ihara extends this definition to depend on a finite dimensional representation of Γ over a field of characteristic 0, although in fact the zeta function so obtained depends only on the character of the given representation. Ihara's earlier definition is identified with the zeta function for the trivial representation of Γ. Ihara's zeta function is similar to the Selberg ζ-function for compact Riemann surfaces.

Ihara's zeta function is defined as an infinite product, but in fact Ihara proves that it is a rational function. Let $Z_\Gamma(u, \chi)$ be the Ihara zeta function with χ the character of the representation of Γ. Then Ihara's rationality formula states:

$$Z_\Gamma(u, \chi) = (1 - u^2)^s \det(I - Au + pu^2) \tag{0.1}$$

where s is an integer and A is a finite matrix coming out of Ihara's structure theorem for Γ. Using the rationality formula, one can count the number of conjugacy classes of primitive elements of Γ. In addition, the Ihara zeta function carries information about the spectral decomposition of the representation of $G = PGL(2, \Bbbk_p)$ on $l^2(\Gamma \backslash G)$.

Date: August 12, 1999.

Serre [14, Introduction] remarked that Ihara's zeta function can be interpreted as a zeta function of certain $(p+1)$-regular finite graphs. In [9], the authors generalize some of Ihara's work in the context of finite graphs. Then, in [7, 8], Hashimoto defines the zeta function of a finite connected graph A as follows: For $\rho : \pi_1(A) \to GL(V_\rho)$ any finite dimensional representation of $\pi_1(A)$, the zeta function of A is defined as

$$Z(\rho, u) = \prod_{\gamma \in P} \det(I - \rho(\gamma) u^{\ell(\gamma)}) \qquad (\in \mathbb{C}[[u]]),$$

where P is the set of free homotopy classes of primitive closed paths in A, and $\ell(\gamma)$ is the shortest edge length for paths in the homotopy class of γ. To prove a rationality formula, Hashimoto assumes that the graph A is bipartite. In the cases that A is regular or bipartite biregular he gives an interpretation for the number s in (0.1) in terms of the Euler characteristic of the graph A. The important new step is to express the zeta function in terms of a rational formula involving edges of the graph. Specifically, Hashimoto defines two operators $T_{1,\rho}$ and $T_{2,\rho}$ on the vector space generated by edges of A and then proves that the zeta function of a A is the determinant of an expression involving $T_{1,\rho}$ and $T_{2,\rho}$.

The zeta function of a uniform tree lattice is defined by H. Bass [2]. A *uniform tree lattice* consists of a tree X and a countable group Γ which acts discretely on X with finite quotient. Bass' work completes the program of defining zeta functions for general finite graphs, and furthermore allows for non-free actions of Γ where the quotient is then a graph of finite groups. To prove the rationality formula, Bass introduces an operator T on oriented edges of X and expresses the zeta function in terms of T, which greatly simplifies the situation.

For a general survey and an elementary introduction to Ihara-type zeta functions, see [15].

In this paper, we define zeta functions for general tree lattices bearing appropriate representations which are allowed to be infinite dimensional representations on Hilbert spaces. This generalization is motivated by an attempt to generalize the work of Bass to define the zeta function associated to a non-uniform tree lattice. This generalization gives rise to a theory of zeta functions for infinite graphs with symmetry. Specifically, for an infinite graph equipped with an action of a discrete group such that the quotient graph has finite volume. In particular one case of interest is that of non-uniform lattices, where Γ and X are as above except that $\Gamma\backslash X$ is infinite but satisfies the finite volume condition:

$$\sum_{x \in \Gamma\backslash X} \frac{1}{|\Gamma_{\tilde{x}}|} < \infty,$$

where $\widetilde{x}$ is any representative for x. Before stating precisely the main results, some background is required.

To deal with infinite graphs, we use the machinery of von Neumann algebras. The model for our techniques originated with the L^2-Betti numbers, defined by Atiyah [1] in 1976. Since then, the use of von Neumann algebras to extend classical topological invariants to non-compact spaces has become fairly well established. For example, see [12] for a recent survey.

A von Neumann algebra $\mathcal{A}$ is a *-algebra of bounded operators on Hilbert space which is closed in the weak (or strong) topology. The von Neumann algebra $\mathcal{A}$ is equipped with a trace function $\mathrm{Tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$. A Hilbert $\mathcal{A}$-module M is a Hilbert space with an $\mathcal{A}$-module structure and an additional projectivity condition. The trace extends to a trace for bounded operators on M which commute with the $\mathcal{A}$ action, and this trace gives rise to a real-valued dimension theory for Hilbert $\mathcal{A}$-modules.

Given a formal power series with coefficients which are trace class operators on a Hilbert $\mathcal{A}$-module, one can use $\mathrm{Exp} \circ \mathrm{Tr}_{\mathcal{A}} \circ \mathrm{Log}$ to define a determinant $\mathrm{Det}_{\mathcal{A}}$ as a formal power series over $\mathbb{C}$. This technique is used in Bass [2]. However, the application to von Neumann algebras is apparently new to this paper and yields a generalization of the classical real-valued determinant of Fuglede and Kadison [6]. We will use the determinant $\mathrm{Det}_{\mathcal{A}}$ to define a zeta function analogous to the zeta function for finite graphs.

Let Γ be a lattice of a tree X which has uniformly bounded valence. Let M be a Hilbert $\mathcal{A}$-module, and ρ a representation from Γ to the unitary operators on M which commute with $\mathcal{A}$.

Let $\varepsilon \in \mathcal{E}(\Gamma)$ be a hyperbolic end of X, that is, ε is one of the two ends of the axis of some hyperbolic element of Γ. For $g \in \Gamma_\varepsilon$(the stabilizer of ε) we can define the (signed) translation distance towards ε, $\tau_\varepsilon(g) \in \mathbb{Z}$. The kernel of τ_ε is $\Gamma^0_\varepsilon < \Gamma_\varepsilon$, and define $\ell(\varepsilon) > 0$ by the equation $\mathrm{Image}(\tau_\varepsilon) = \ell(\varepsilon)\mathbb{Z}$. Put

$$\sigma_\varepsilon = \frac{1}{|\Gamma^0_\varepsilon|} \sum_{\substack{g \in \Gamma_\varepsilon \\ \tau_\varepsilon(g) = \ell(\varepsilon)}} g$$

Definition. For Γ an X-lattice and ρ a unitary Hilbert $\mathcal{A}$-representation of Γ, define the zeta function of Γ by

$$Z_\rho(u) = \prod_{\varepsilon \in \Gamma \backslash \mathcal{E}(\Gamma)} \mathrm{Det}_{\mathcal{A}}(I - \rho(\sigma_\varepsilon) u^{\ell(\varepsilon)}) \quad \in \mathbb{C}[[u]] \tag{0.2}$$

This zeta function is formally quite similar to that of Bass [2]. One crucial difference is that for a non-uniform lattice there may be infinitely many classes $\varepsilon \in \Gamma\backslash\mathcal{E}(\Gamma)$ with the same length, and so it is not even clear that the coefficients of $Z_\rho(u)$ converge. In fact, conditions on ρ are needed, and will be stated precisely in Theorem 0.1 to follow.

Let $H_\rho(EX)$ denote the space of Γ-equivariant functions f from the edges of X to M, which satisfy

$$\|f\|^2_\rho = \sum_{e \in \Gamma\backslash EX} \|f(\tilde{e})\|^2_M < \infty.$$

Define the operator $T_\rho : H_\rho(EX) \to H_\rho(EX)$ by

$$T_\rho f(e) = \sum_{(e,\, e_1) \text{ reduced}} f(e_1).$$

The first theorem of this paper relates the zeta function to the determinant of T_ρ.

Theorem 0.1 (Bass-Hashimoto Formula)**.** *Suppose X, Γ, and ρ satisfy the two finiteness conditions:*

1. $\sum_{e \in \Gamma\backslash EX} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{e}}} < \infty$.

2. *For all n,*

$$\sum_{\substack{\varepsilon\in\Gamma\backslash\mathcal{E}(\Gamma);\\ \ell(\varepsilon)=n}} \dim_{\mathcal{A}} M^{\Gamma^0_\varepsilon} < \infty.$$

Then $Z_\rho(u)$ is well defined in $\mathbb{C}[[u]]$, and

$$Z_\rho(u) = \mathrm{Det}_{\mathcal{A}}(I - T_\rho u). \tag{0.3}$$

In addition, the power series $Z_\rho(u)$ converges for $u \in \mathbb{C}$, $|u| < \|T_\rho\|$.

The two finiteness conditions can intuitively be interpreted as saying that the total volume of the edges of $\Gamma\backslash X$ is finite, and the total volume of loops of length n in $\Gamma\backslash X$ is finite for each n. The subtlety is that even if the stabilizers $\Gamma_{\tilde{e}}$, Γ^0_ε are large, the invariant subspaces $M^{\Gamma_{\tilde{e}}}$ and $M^{\Gamma^0_\varepsilon}$ may not be small.

The next result relates the zeta function to a Laplace operator on the vertices of X. We define $H_\rho(VX)$ analogous to $H_\rho(EX)$ but using Γ-invariant functions on the vertices of X. Put

$$\begin{aligned}\chi_\rho(X) &= \dim_{\mathcal{A}} H_\rho(VX) - \frac{1}{2}\dim_{\mathcal{A}} H_\rho(EX)\\ &= \sum_{x\in VB} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{x}}} - \frac{1}{2}\sum_{e\in EB} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{e}}}.\end{aligned}$$

Define the operators $\delta_\rho, Q_\rho : H_\rho(VX) \to H_\rho(VX)$ by

$$\delta_\rho f(x) = \sum_{\partial_0 e = x} f(\partial_1 e) \tag{0.4}$$

$$Q_\rho f(x) = (\deg(x) - 1) f(x). \tag{0.5}$$

Here δ_ρ is the usual adjacency operator.

Theorem 0.2 (Bass-Hashimoto-Ihara Formula). *When $\sum_{e\in EB} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{e}}} < \infty$,*

$$\mathrm{Det}_{\mathcal{A}}(I - T_\rho u) = (1 - u^2)^{-\chi_\rho(X)}\, \mathrm{Det}_{\mathcal{A}}(I - \delta_\rho u + Q_\rho u^2)$$

where exponentiation is defined formally using the principal branch of the logarithm.

The remainder of the paper consists of interpreting Theorem 0.1 and Theorem 0.2 for important special cases.

Let $1 \to \Lambda \to \Gamma \to \pi \to 1$ be an exact sequence. The most interesting examples occur when $\mathcal{A}$ is the von Neumann algebra of the quotient group $\pi = \Gamma/\Lambda$, Λ acts freely on X, and ρ is the coset representation of Γ on $M = l^2(\pi)$. In this situation, we may interpret $Z_\rho(u)$ to be the π-zeta function of the infinite graph $Y = \Lambda\backslash X$:

Theorem 0.3. *Suppose Y is a locally finite graph and the discrete group π acts on Y with quotient B having finite volume. Let $\Delta_u = I - \delta u + Qu^2$ acting on $l^2(VY)$. Then*

$$Z_\pi(Y, u) = \prod_{\gamma\in\pi\backslash P} \left(1 - u^{\ell(\gamma)}\right)^{\frac{1}{|\pi_\gamma|}} = (1 - u^2)^{-\chi^{(2)}(Y)}\, \mathrm{Det}_\pi(\Delta_u).$$

Here P is the set of free homotopy classes of primitive closed paths in Y. For $\gamma \in P$, $\ell(\gamma)$ is the length of the shortest representative of γ. The group π_γ is the *stabilizer of γ under the action of π. The Euler characteristic $\chi_\rho(X)$ is the L^2 Euler characteristic of $\chi^{(2)}(Y)$ as in* [4], *and can be computed as*

$$\chi^{(2)}(Y) = \sum_{x \in VB} \frac{1}{|\pi_x|} - \frac{1}{2} \sum_{e \in EB} \frac{1}{|\pi_e|}.$$

The structure of this paper is as follows. Section one is a review of graphs and group actions on graphs. Section two is a treatment of von Neumann algebras and Hilbert modules, the main novelty of which is our definition of a complex valued von Neumann determinant.

In the third section, we introduce the zeta function and prove Theorem 0.1 and Theorem 0.2. Section four specializes to the main example of a coset representation and proves Theorem 0.3. Finally, Section five presents explicit examples.

1. Groups and Trees

This section is a review of graphs and group actions on graphs, and mainly consists of definitions and notation to be used later.

Definition 1.1. A *graph* $A = (VA, EA)$ consists of a set of vertices VA, a set of edges EA, maps ∂_0, $\partial_1 : EA \to VA$, and a fixed point free involution $\bar{}: EA \to EA$ for which $\partial_i \bar{e} = \partial_{1-i} e$ for all $e \in EA$.

For a graph A and $a \in VA$, let $E_0(a) = \partial_0^{-1}(a) = \{e \in EA \mid \partial_0 e = a\}$. A is called *k-regular* if $|E_0(a)| = k$ for all $a \in VA$, where $|E_0(a)|$ is the *valence* (degree) of the vertex a. A graph is *regular* if it is k-regular for some k.

The sequence $P = (e_1, e_2, \ldots, e_n)$ is a *path* of length n from $\partial_0 e_1$ to $\partial_1 e_n$ in A if $e_i \in EA$ and $\partial_1 e_i = \partial_0 e_{i+1}$, $1 \le i \le n-1$. For $a_0 \in A$, the empty path from a_0 to a_0 is denoted (a_0) and has length 0. A path P is *closed* if $\partial_0 e_1 = \partial_1 e_n$ or if P has length 0. P is *reduced* if $e_{i+1} \ne \bar{e}_i$, $1 \le i \le n-1$ or if P has length 0.

A nonempty graph A is *connected* if any two vertices of A can be joined by a path. A graph is a *tree* if it is connected and every reduced path with positive length is not closed. A graph is *locally finite* if the degree of each vertex is finite. A tree has *bounded degree* if there is a uniform bound d on the degrees of vertices.

Let A and B be graphs. A *morphism* $f : A \to B$ is a map taking $VA \to VB$, $EA \to EB$ which commutes with the involution and with ∂_i. A morphism is an *automorphism* if it also induces a bijection between the vertices and edges of A and B. An automorphism of a graph is called an *inversion* if it takes some edge e to $\bar{e}$.

Let $G \le \mathrm{Aut}(A)$ where A is a connected graph and $\mathrm{Aut}(A)$ is the group of automorphisms of A. Assume G does not contain any inversion. The quotient set $G \backslash A$ naturally has the structure of a graph.

Now let X be a locally finite tree and $G = \mathrm{Aut}(X)$ the group of automorphisms of X. Then G is a locally compact group with the compact open topology coming from the action of G on X (X has the discrete topology). The stabilizers of vertices of X are compact and open. The group $\Gamma \le G$ is discrete if and only if Γ_x is finite for some (hence for all) vertices $x \in X$.

Assume that Γ does not contain any inversions. As in [3],

Definition 1.2. The *volume* of $\Gamma \backslash X$ is defined to be

$$\mathrm{vol}(\Gamma \backslash X) = \sum_{x \in \Gamma \backslash X} \frac{1}{|\Gamma_{\tilde{x}}|}$$

where $\tilde{x}$ is any representative of x. When $\mathrm{vol}(\Gamma\backslash X) < \infty$, say that Γ has *finite covolume* and that Γ is an *X-lattice*. In particular Γ is called *uniform* if $\Gamma\backslash X$ is finite and *non-uniform* otherwise. One can similarly define an *edge-volume* by

$$\mathrm{vol}(\Gamma\backslash EX) = \sum_{e\in\Gamma\backslash EX} \frac{1}{|\Gamma_{\tilde{e}}|}$$

It is not hard to show that for a bounded degree tree, $\mathrm{vol}(\Gamma\backslash X) < \infty$ if and only if $\mathrm{vol}(\Gamma\backslash EX) < \infty$.

If Γ is an X-lattice it is also a lattice of $\mathrm{Aut}(X)$ which implies that $\mathrm{Aut}(X)$ is unimodular, *i.e.* the left Haar measure is also a right Haar measure.

2. Von Neumann Algebras

The material in this section is mostly classical, although some of Section 2.3 is new. References for this material are [5] and [13].

2.1. Traces on von Neumann algebras.

Let H be a separable complex Hilbert space, and let $B(H,H)$ denote the C^*-algebra of bounded linear operators on H. In the *weak topology* on $B(H,H)$, $T_n \to T$ if and only if $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in H$. In the *strong topology* on $B(H,H)$, $T_n \to T$ if and only if $\|T_n x\| \to \|Tx\|$ for all $x \in H$.

Definition 2.1. A *von Neumann algebra* is a subalgebra $\mathcal{A} \leq B(H,H)$ which is closed under adjoints and such that $\mathcal{A}$ is closed in the weak operator topology.

For a subset $M \subset B(H,H)$, the *commutant* of M is $M' = \{T \in B(H,H) \mid ST = TS, \forall S \in M\}$. Clearly $M \subset M''$.

The double commutant theorem, due to von Neumann, states that the conditions

1. $\mathcal{A}'' = \mathcal{A}$
2. $\mathcal{A}$ is weakly closed
3. $\mathcal{A}$ is strongly closed

are all equivalent.

An element $T \in \mathcal{A}$ is *positive* if T is self-adjoint and $\langle Tf, f\rangle \geq 0$ for all $f \in H$. Let $\mathcal{A}^+$ denote the cone of positive elements. A map $\mathrm{Tr} : \mathcal{A}^+ \to [0,\infty]$ is called a *trace* if $\mathrm{Tr}(S+T) = \mathrm{Tr}(S) + \mathrm{Tr}(T)$, $\mathrm{Tr}(\lambda T) = \lambda \mathrm{Tr}(T)$, and $\mathrm{Tr}(TT^*) = \mathrm{Tr}(T^*T)$ for all $S, T \in \mathcal{A}$, $\lambda \in [0,\infty)$. A trace is *finite* if it always assumes finite values. A trace is *faithful* if $\mathrm{Tr}(T) = 0$ implies $T = 0$ for $T \in \mathcal{A}^+$. A trace is *normal* if for any monotone increasing net $\{T_i\}_{i\in I}$ ($T_i - T_j$ is positive for $i > j$) with $T = \sup T_i$, one has $\mathrm{Tr}\, T = \sup \mathrm{Tr}\, T_i$.

A von Neumann algebra is called *finite* if it has a finite, faithful, normal trace. A finite trace extends uniquely to a $\mathbb{C}$-linear map $\mathrm{Tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$, which satisfies $\mathrm{Tr}_{\mathcal{A}} ST = \mathrm{Tr}_{\mathcal{A}} TS$ for all $S, T \in \mathcal{A}$. From now on, all von Neumann algebras will be finite and equipped with corresponding trace.

The trace on $\mathcal{A}$ gives an inner product on $\mathcal{A}$ defined by $\langle S, T\rangle = \mathrm{Tr}(T^*S)$. Let $\hat{\mathcal{A}}$ denote the Hilbert completion of $\mathcal{A}$. $\mathcal{A}$ acts on the left on $\hat{\mathcal{A}}$, and we let $B^{\mathcal{A}}(\hat{\mathcal{A}})$ denote the space of bounded, $\mathcal{A}$-equivariant operators on $\hat{\mathcal{A}}$. The opposite algebra $\mathcal{A}^{op}$ acts on the right on $\hat{\mathcal{A}}$, and the right representation $R : \mathcal{A}^{op} \to B^{\mathcal{A}}(\hat{\mathcal{A}})$ is an isomorphism (see [5] for a proof of this fundamental result). Therefore the trace on $\mathcal{A}$ gives rise to a trace on $B^{\mathcal{A}}(\hat{\mathcal{A}})$, also denoted by $\mathrm{Tr}_{\mathcal{A}}$.

Example 2.2. One can take $\mathcal{A} = \mathbb{C} \subset B(H,H)$, and set $\mathrm{Tr}_{\mathcal{A}}\, c = c$.

Example 2.3. Suppose π is a countable discrete group. The group ring $\mathbb{C}[\pi]$ acts on the left on $l^2(\pi)$. The von Neumann algebra $\mathcal{N}(\pi)$ is the von Neumann algebra generated by $\mathbb{C}[\pi]$, that is, the weak closure of $\mathbb{C}[\pi]$. For $a = \sum_{g\in\pi} a_g \cdot g \in \mathcal{N}(\pi)$, the trace is given by

$$\operatorname{Tr}_\pi(a) = a_{1_\pi}.$$

2.2. **Hilbert modules.**

Definition 2.4. A *Hilbert module* for $\mathcal{A}$ is a Hilbert space M with a continuous left $\mathcal{A}$-module structure, and such that there exists a Hilbert space H and an $\mathcal{A}$-equivariant isometric embedding ι of M onto a closed subspace of the Hilbert space $\hat{\mathcal{A}}\hat{\otimes}H$. Here $\hat{\otimes}$ denotes the completed tensor product, and $\mathcal{A}$ acts trivially on H. Also note that H and ι must exist, but are not part of the structure.

$B^{\mathcal{A}}(\hat{\mathcal{A}}\hat{\otimes}H)$ is identified with the algebra $\mathcal{A}^{op}\hat{\otimes}B(H)$. If $(x_1, x_2, \dots)$ is an orthonormal basis for H, $T \in B^{\mathcal{A}}(\hat{\mathcal{A}}\hat{\otimes}H)$ is identified with a matrix with entries in $\mathcal{A}$ acting by right multiplication. More precisely, let $P_j : \hat{\mathcal{A}}\hat{\otimes}H \to \hat{\mathcal{A}}$ be the j^{th} coordinate projection, given by $P_j(a \otimes y) = \langle y, x_j\rangle\, a$. Then

$$T_{ij}(a) = P_i T(a \otimes x_j) : \hat{\mathcal{A}} \to \hat{\mathcal{A}}.$$

Say that T is $\mathcal{A}$-*Hilbert-Schmidt* if $\sum_{i,j} \operatorname{Tr}_{\mathcal{A}}(T_{ij}^* T_{ij}) < \infty$. Say that T is of $\mathcal{A}$-*trace class* if T is the product of two $\mathcal{A}$-Hilbert-Schmidt operators. If T is $\mathcal{A}$-trace class, the von Neumann trace of T is $\sum_{i=1,2,\dots} \operatorname{Tr}_{\mathcal{A}} T_{ii}$. As usual, this does not depend on the basis of H.

If M is a Hilbert module then orthogonal projection onto M, $P_M : \hat{\mathcal{A}}\hat{\otimes}H \to \hat{\mathcal{A}}\hat{\otimes}H$, commutes with $\mathcal{A}$. Let

$$\dim_{\mathcal{A}} M = \operatorname{Tr}_{\mathcal{A}} P_M.$$

This is well defined, *i.e.* $\dim_{\mathcal{A}} M$ is independent of the choice of H and the embedding ι. For a bounded operator $T : M \to M$ commuting with $\mathcal{A}$, define $\operatorname{Tr}_{\mathcal{A}} T = \operatorname{Tr}_{\mathcal{A}}(P_M \circ T \circ \iota_M)$ whenever the right hand side is of $\mathcal{A}$-trace class.

It is not hard to show that if $P : M \to M$ is a projection and T is bounded, then

$$|\operatorname{Tr}_{\mathcal{A}} PT| \leq \|T\| \dim_{\mathcal{A}} \operatorname{Image}(P), \tag{2.1}$$

where $\|T\|$ is the operator norm. In particular if $\dim_{\mathcal{A}} M < \infty$ then for any bounded $T : M \to M$, $\operatorname{Tr}_{\mathcal{A}} T$ is defined and in fact

$$|\operatorname{Tr}_{\mathcal{A}} T| \leq \|T\| \dim_{\mathcal{A}} M. \tag{2.2}$$

Example 2.5. Let $\mathcal{A} = \mathbb{C} \subset B(\mathbb{C}, \mathbb{C})$ as in Example 2.2. Then $\mathcal{A}$-Hilbert modules are simply $\mathbb{C}$-vector spaces, we arrive at the standard definitions of trace class, and Tr_A and $\dim_{\mathcal{A}}$ are the ordinary trace and dimension.

2.3. **Determinants.** This section defines a notion of determinant for Hilbert modules. Let M be a finite dimensional Hilbert $\mathcal{A}$-module. The determinant is defined for formal power series in u with coefficients in $\mathbf{B}_M = B^{\mathcal{A}}(M)$, using methods similar to [2]. These determinants are formal power series over $\mathbb{C}$. In our applications the series converge for small $u \in \mathbb{C}$ to a complex-valued determinant which generalizes the real-valued Fuglede-Kadison determinant defined in [6].

Let u be an indeterminant. For a ring R let $R[[u]]$ denote the ring of formal power series in u with coefficients in R, and let $R^+[[u]]$ denote those power series with zero constant term.

Because M is finite dimensional,

$$\operatorname{Tr}_{\mathcal{A}} : \mathbf{B}_M \to \mathbb{C}$$

is well defined and satisfies (2.1) for all $T \in \mathbf{B}_M$. The trace on $\mathbf{B}_M$ extends to $\operatorname{Tr}_{\mathcal{A}} : \mathbf{B}_M[[u]] \to \mathbb{C}[[u]]$ by applying to each coefficient.

The exponential $\operatorname{Exp} : \mathbf{B}_M[[u]] \to \operatorname{Id} + \mathbf{B}_M^+[[u]]$ and the principal branch of logarithm $\operatorname{Log} : \operatorname{Id} + \mathbf{B}_M^+[[u]] \to \mathbf{B}_M[[u]]$ are defined as series. Here Id is the identity operator on M. Exp and Log are mutually inverse (see [2] and references therein). Define $\operatorname{Det}_{\mathcal{A}} : \operatorname{Id} + \mathbf{B}_M^+[[u]] \to \operatorname{Id} + \mathbb{C}^+[[u]]$ so that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Id} + \mathbf{B}_M^+[[u]] & \xrightarrow{\operatorname{Det}_{\mathcal{A}}} & 1 + \mathbb{C}^+[[u]] \\ {\scriptstyle\operatorname{Log}}\downarrow & & {\scriptstyle\operatorname{Log}}\downarrow \\ \mathbf{B}_M^+[[u]] & \xrightarrow{\operatorname{Tr}_{\mathcal{A}}} & \mathbb{C}^+[[u]] \end{array}$$

Proposition 2.6. *The determinant* $\operatorname{Det}_{\mathcal{A}}$ *satisfies the following:*

1. *For* $\mathbf{c} \in 1 + \mathbb{C}^+[[u]]$, $\operatorname{Det}_{\mathcal{A}}(\mathbf{c}\operatorname{Id}) = \mathbf{c}^{\dim_{\mathcal{A}} M} = \operatorname{Exp}(\dim_{\mathcal{A}} M \operatorname{Log} \mathbf{c})$.
2. *For* $\boldsymbol{\alpha} \in \operatorname{Id} + \mathbf{B}_M^+[[u]]$ *and invertible* $\boldsymbol{\beta} \in \mathbf{B}_M[[u]]$, $\operatorname{Det}_{\mathcal{A}}(\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}^{-1}) = \operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha})$.
3. *For* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \operatorname{Id} + \mathbf{B}_M^+[[u]]$, $\operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha}\boldsymbol{\beta}) = \operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha}) \operatorname{Det}_{\mathcal{A}}(\boldsymbol{\beta})$.
4. *Let* H_0, H_1 *be finite dimensional Hilbert* $\mathcal{A}$*-modules. Then* $B^{\mathcal{A}}(H_0 \oplus H_1)$ *can be identified with matrices in the form*
$$\alpha = \begin{bmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{bmatrix} \in \begin{bmatrix} B^{\mathcal{A}}(H_0, H_0) & B^{\mathcal{A}}(H_1, H_0) \\ B^{\mathcal{A}}(H_0, H_1) & B^{\mathcal{A}}(H_1, H_1) \end{bmatrix}.$$
Let $\boldsymbol{\alpha} \in \operatorname{Id} + \mathbf{B}_{H_0 \oplus H_1}^+[[u]]$ *be*
$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_{00} & * \\ 0 & \boldsymbol{\alpha}_{11} \end{bmatrix}$$
with $\boldsymbol{\alpha}_{ii} \in \mathbf{B}_{H_i}[[u]]$*. Then*
$$\operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha}) = \operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha}_{00}) \operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha}_{11}).$$

Proof. 1. $\operatorname{Log}(\operatorname{Det}_{\mathcal{A}}(\mathbf{c}\operatorname{Id})) = \operatorname{Tr}_{\mathcal{A}} \operatorname{Log}(\mathbf{c}\operatorname{Id}) = \operatorname{Tr}_{\mathcal{A}}(\operatorname{Id}\operatorname{Log}(\mathbf{c})) = \dim_{\mathcal{A}} M \operatorname{Log}(\mathbf{c})$.
2. $\operatorname{Log}\operatorname{Det}_{\mathcal{A}}(\boldsymbol{\beta}\boldsymbol{\alpha}\boldsymbol{\beta}^{-1}) = \operatorname{Tr}_{\mathcal{A}}(\boldsymbol{\beta}\operatorname{Log}(\boldsymbol{\alpha})\boldsymbol{\beta}^{-1}) = \operatorname{Tr}_{\mathcal{A}} \operatorname{Log}(\boldsymbol{\alpha}) = \operatorname{Log}\operatorname{Det}_{\mathcal{A}}(\boldsymbol{\alpha})$.
3. This follows from the Campbell-Baker-Hausdorff formula and the fact that $\operatorname{Tr}_{\mathcal{A}}$ vanishes on commutators.
4. This follows from $\operatorname{Tr}_{\mathcal{A}} \alpha = \operatorname{Tr}_{\mathcal{A}} \alpha_{00} + \operatorname{Tr}_{\mathcal{A}} \alpha_{11}$.

See [2, I-4] for details. ∎

Proposition 2.7. *Fix* $T \in B^{\mathcal{A}}(M)$*. Suppose* $u \in \mathbb{C}$ *satisfies* $|u| < 1/\|T\|$*.*

1. *The series* $\operatorname{Det}_{\mathcal{A}}(I - uT)$ *converges absolutely.*
2. $|\operatorname{Det}_{\mathcal{A}}(I - uT)| = \operatorname{Det}_{\mathcal{A}}^{FK}(I - uT)$*, where the right hand side is the Fuglede-Kadison* [6] *determinant.*

Proof.

$$\operatorname{Det}_{\mathcal{A}}(I - uT) = \operatorname{Exp}\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}_{\mathcal{A}}(uT)^n\right).$$

The sum converges uniformly since by (2.1), $|\operatorname{Tr}_{\mathcal{A}}(uT)^n| \le \|uT\|^n \dim_{\mathcal{A}} M$, and since $\|uT\| < 1$.

Now, recall that if $X \in B^{\mathcal{A}}(M)$ is invertible, the Fuglede-Kadison determinant of X is a positive real defined by

$$\log \operatorname{Det}_{\mathcal{A}}^{FK}(X) = \frac{1}{2} \operatorname{Tr}_{\mathcal{A}} \log(X^*X),$$

where $\log(X^*X)$ is defined using the functional calculus for self-adjoint operators. Now we have

$$\begin{aligned}\log|\operatorname{Det}_{\mathcal{A}}(I-uT)|^2 &= \log \operatorname{Det}_{\mathcal{A}}(I-uT)\overline{\operatorname{Det}_{\mathcal{A}}(I-uT)} \\ &= \log \operatorname{Det}_{\mathcal{A}}(I-uT) \operatorname{Det}_{\mathcal{A}}(I-\bar{u}T^*) \\ &= \log \operatorname{Det}_{\mathcal{A}}(I-uT)(I-\bar{u}T^*) \\ &= \operatorname{Tr}_{\mathcal{A}} \log(I-uT)(I-\bar{u}T^*) \\ &= \log(\operatorname{Det}_{\mathcal{A}}^{FK}(I-uT))^2.\end{aligned}$$

□

2.4. **Hilbert representations.** Let Γ be any countable discrete group (although one could allow more general groups), and $\mathcal{A}$ a finite von Neumann algebra.

Definition 2.8. A *Hilbert $\mathcal{A}$-representation* of Γ consists of a Hilbert $\mathcal{A}$-module M and a homomorphism $\rho : \Gamma \to B^{\mathcal{A}}(M)$.

Say that ρ is *unitary* if $\rho(g)$ is a unitary operator for all $g \in \Gamma$. The direct sum of two Hilbert representations is defined in the usual sense. One can form the tensor product over $\mathbb{C}$ of a Hilbert representation with a finite dimensional representation of Γ to obtain a new Hilbert representation.

For $G < \Gamma$, let

$$M^G = \{x \in M \mid \rho(g)x = x, \forall g \in G\}$$

denote the invariant subspace of M under G. Since for $a \in \mathcal{A}$, $g \in G$, and $x \in M^G$ we have $\rho(g)ax = a\rho(g)x = ax$, M^G is a Hilbert submodule of M.

2.5. **Group actions.** Suppose now that Γ acts on a discrete countable set X, and suppose the stabilizers Γ_x are finite for all $x \in X$. Set $B = \Gamma\backslash X$. Let $\rho : \Gamma \to B^{\mathcal{A}}(M)$ be a unitary Hilbert representation. In this section, we define a Hilbert $\mathcal{A}$-module analogous to the space of M-valued l^2 functions on B, but which is more natural when Γ does not act freely.

For Γ-equivariant $f : X \to M$ define

$$\|f\|_\rho^2 = \sum_{x \in B} \|f(\widetilde{x})\|_M^2$$

where for each x we choose any lift $\widetilde{x} \in X$. The definition is independent of the choice of lifts because $\rho(\gamma)$ is unitary for all $\gamma \in \Gamma$.

Now define

$$H_\rho(X) = \{f : X \to M \mid f \text{ is } \Gamma\text{-equivariant, and } \|f\|_\rho < \infty\}.$$

$H_\rho(X)$ is a Hilbert space with norm $\|\cdot\|_\rho$ and inner product

$$\langle f, g\rangle_\rho = \sum_{x \in B} \langle f(\widetilde{x}), g(\widetilde{x})\rangle_M .$$

$H_\rho(X)$ admits an $\mathcal{A}$ action given by

$$(a \cdot f)(x) = a \cdot f(x)$$

where $a \in \mathcal{A}$, $f \in H_\rho(X)$, and $x \in X$. An easy check shows that the action preserves Γ-equivariance of f.

Now choose a fixed lift $\widetilde{x} \in X$ for each $x \in B$. Define $\Phi : H_\rho(X) \to M\hat{\otimes} l^2(B)$ by

$$\Phi(f) = \sum_{x \in B} f(\widetilde{x}) \otimes \delta_x$$

where $\delta_x \in l^2(B)$ is 1 at x and 0 elsewhere. Since the terms $f(\widetilde{x}) \otimes \delta_x$ are mutually orthogonal, Φ is an isometric embedding. This shows that $H_\rho(X)$ is a Hilbert $\mathcal{A}$-module. One should note that the embedding Φ depends on the choice of lifts.

Proposition 2.9. *The von Neumann dimension of $H_\rho(X)$ is given by*

$$\dim_{\mathcal{A}} H_\rho(X) = \sum_{x \in B} \dim_{\mathcal{A}} M^{\Gamma_{\widetilde{x}}}. \tag{2.3}$$

Proof. Fix $x \in B$, and recall that $\Gamma_{\widetilde{x}}$ denotes the stabilizer in Γ of $\widetilde{x} \in X$. For $\phi \in M^{\Gamma_{\widetilde{x}}}$, there is a function $f_\phi \in H_\rho(X)$ supported only on the orbit of $\widetilde{x}$,

$$f_\phi(\gamma\widetilde{x}) = \rho(\gamma)\phi \tag{2.4}$$

It is easy to check that the correspondence $\phi \leftrightarrow f_\phi$ induces an isomorphism of $\mathcal{A}$-Hilbert modules

$$\hat{\bigoplus_{x \in B}} M^{\Gamma_{\widetilde{x}}} \xrightarrow{\sim} H_\rho(X).$$

This proves the Proposition, because the trace on $\mathcal{A}$ is normal. ◻

3. Zeta Functions

3.1. **Tree lattices.** Let X be a locally finite tree, Γ an X-lattice (or a discrete subgroup of $\mathrm{Aut}(X)$ without inversion). Put $B = \Gamma\backslash X$. Notation introduced in this section will be used throughout the rest of this paper.

We denote the path metric on X by d. It is clear that Γ induces an action on the ends of X, where an *end* is a class of reduced infinite paths in X which eventually coincide.

For $g \in \Gamma$ let $\ell(g) = \min_{x \in X} d(gx, x)$ and let $X_g = \{x \in X \mid d(gx, x) = \ell(g)\}$. Notice that X_g is the smallest subgraph of X which is invariant under the action of $\langle g \rangle$. If $\ell(g) = 0$ (g is *elliptic*), then g is of finite order and X_g is the tree of fixed points of g. We call g *hyperbolic* if $\ell(g) > 0$. In this case X_g is an infinite ray, isomorphic to $\mathbb{Z}$, which we call the *axis* of g. The induced action of g on X_g is just a translation of amplitude $\ell(g)$ towards one of the two ends of X_g, denoted by $\varepsilon(g)$. When g is hyperbolic $\varepsilon(g)$ and $\varepsilon(g^{-1})$ are the only two ends of X fixed by g.

Let

$$\mathcal{E} = \mathcal{E}(\Gamma) = \{\varepsilon(g) \mid g \in \Gamma, \ell(g) > 0\} \tag{3.1}$$

be the set of hyperbolic ends of X, and put $\mathcal{P} = \mathcal{P}(\Gamma) = \Gamma\backslash\mathcal{E}(\Gamma)$.

For $\varepsilon \in \mathcal{E}(\Gamma)$ let Γ_ε be the stabilizer of ε. Let $[x, \varepsilon)$ be the infinite ray from x towards the end ε. Define $\tau_\varepsilon : \Gamma_\varepsilon \to \mathbb{Z}$ by

$$\tau_\varepsilon(\gamma) = d(x, u) - d(\gamma x, u) \tag{3.2}$$

where $x \in X$ and $u \in [x, \varepsilon) \cap [gx, \varepsilon)$. The function τ_ε is a homomorphism [2]. For $\gamma \in \Gamma_\varepsilon$, $|\tau_\varepsilon(\gamma)| = \ell(\gamma)$. In particular, if $\varepsilon = \varepsilon(g)$ then $\tau_\varepsilon(g) = \ell(g)$. Put

$$\Gamma_\varepsilon^0 = \mathrm{Ker}(\tau_\varepsilon)$$

and define $\ell(\varepsilon) > 0$ so that

$$\ell(\varepsilon)\mathbb{Z} = \mathrm{Image}(\tau_\varepsilon).$$

For $\varepsilon = \varepsilon(g) \in \mathcal{E}(\Gamma)$ let $X_\varepsilon = X_g$. This definition is independent of g because all hyperbolic elements associated to the same end have the same axis [2, II-2.4]. The graph X_ε is invariant under Γ_ε, Γ_ε^0 is finite and acts trivially on X_ε, and the graph $\Gamma_\varepsilon \backslash X_\varepsilon$ is a cycle of length $\ell(\varepsilon)$.

3.2. **Zeta functions of uniform tree lattices.** This section describes the zeta function associated to a uniform lattice, due to Bass [2].

Suppose $\rho : \Gamma \to GL(V_\rho)$ is a finite dimensional $\mathbb{C}$-representation of Γ. For $\varepsilon \in \mathcal{E}(\Gamma)$ put

$$\sigma_\varepsilon = \frac{1}{|\Gamma_\varepsilon^0|} \sum_{\substack{g \in \Gamma_\varepsilon \\ \tau_\varepsilon(g) = \ell(\varepsilon)}} g \tag{3.3}$$

The zeta function associated to the action of Γ on X and the representation ρ is defined as

$$Z_\rho(u) = \prod_{\varepsilon \in \mathcal{P}} \mathrm{Det}\left(\rho(1_\Gamma - \sigma_\varepsilon u^{\ell(\varepsilon)})\right). \tag{3.4}$$

For this section only, let $H_\rho(EX)$ (respectively $H_\rho(VX)$) be the space of Γ-equivariant functions from EX (respectively VX) to V_ρ. Define the operators $T_\rho : H_\rho(EX) \to H_\rho(EX)$ and $\delta_\rho, Q_\rho : H_\rho(VX) \to H_\rho(VX)$ by

$$T_\rho f(e) = \sum_{(e, e_1) \text{ reduced}} f(e_1) \tag{3.5}$$

$$\delta_\rho f(x) = \sum_{\partial_0 e = x} f(\partial_1 e) \tag{3.6}$$

$$Q_\rho f(x) = q(x) f(x) \tag{3.7}$$

where $q(x) + 1$ is the degree of x. The operator δ_ρ is known as the *adjacency operator*.

Bass' theorem [2] says that $Z_\rho(u) = \mathrm{Det}(I - T_\rho u)$, which results in a generalization of Ihara's theorem:

$$Z_\rho(u) = (1 - u^2)^{-\chi_\rho(\Gamma \backslash X)} \mathrm{Det}(I - \delta_\rho u + Q_\rho u^2),$$

where

$$\chi_\rho(\Gamma \backslash X) = \sum_{x \in \Gamma \backslash X} \dim V_\rho^{\Gamma_{\tilde{x}}} - \frac{1}{2} \sum_{e \in \Gamma \backslash EX} \dim V_\rho^{\Gamma_{\tilde{e}}}$$

is the Euler characteristic of $\Gamma \backslash X$ corresponding to ρ.

3.3. **Zeta functions of general tree lattices.** Suppose X is a locally finite tree with a uniform bound d on the degree of vertices. In this section we define the zeta function associated to an X-lattice Γ and a Hilbert representation of Γ which satisfies certain finiteness conditions. In fact the theory applies to any discrete group $\Gamma < \mathrm{Aut}(X)$ without inversion, equipped with a representation which satisfies the finiteness conditions. However, we concentrate only on the case where Γ is a lattice.

Let $\mathcal{A}$ be a finite von Neumann algebra with trace $\mathrm{Tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$. Let M be any Hilbert $\mathcal{A}$-module, and $\rho : \Gamma \to B^{\mathcal{A}}(M)$ a unitary Hilbert representation.

Using Section 2.5, we have Hilbert $\mathcal{A}$-modules $H_\rho(EX)$ and $H_\rho(VX)$. By (2.3),

$$\dim_{\mathcal{A}} H_\rho(EX) = \sum_{e \in EB} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{e}}} \tag{3.8}$$

$$\dim_{\mathcal{A}} H_\rho(VX) = \sum_{x \in VB} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{x}}}. \tag{3.9}$$

Using these spaces, define the operators T_ρ, Q_ρ and δ_ρ as in (3.5).

Define

$$\mathcal{E}_n = \{\varepsilon \in \mathcal{E} \mid \ell(\varepsilon) \text{ divides } n\}.$$

The action of Γ on $\mathcal{E}$ preserves length, so Γ also acts on $\mathcal{E}_n$. Set $\mathcal{P}_n = \mathcal{P}_n(\Gamma) = \Gamma \backslash \mathcal{E}_n$.

For $\varepsilon \in \mathcal{E}$, let X_ε^+ denote the set of edges of X_ε which are oriented towards ε. Then define

$$\begin{aligned} \mathcal{E}_n(E) &= \{(\varepsilon, e) \mid \varepsilon \in \mathcal{E}_n, e \in X_\varepsilon^+\} \\ \mathcal{E}_n(e) &= \{\varepsilon \mid \varepsilon \in \mathcal{E}_n, e \in X_\varepsilon^+\}. \end{aligned}$$

There is a natural diagonal action of Γ on $\mathcal{E}_n(E)$, where $\gamma \in \Gamma$ sends $(\varepsilon, e) \in \mathcal{E}_n(E)$ to $(\gamma\varepsilon, \gamma e)$.

Lemma 3.1. *With $H_\rho(\mathcal{E}_n(E))$ defined as in Section 2.5,*

$$\dim_{\mathcal{A}} H_\rho(\mathcal{E}_n(E)) = \sum_{\varepsilon \in \mathcal{P}_n} \ell(\varepsilon) \dim_{\mathcal{A}} M^{\Gamma_\varepsilon^0}.$$

Proof. Stabilizers in Γ of $(\varepsilon, e) \in \mathcal{E}_n(E)$ must fix ε and must have translation length 0 to fix e. That is, $\Gamma_{(\varepsilon,e)} = \Gamma_\varepsilon^0$. Using (2.3),

$$\begin{aligned} \dim_{\mathcal{A}} H_\rho(\mathcal{E}_n(E)) &= \sum_{x \in \Gamma \backslash \mathcal{E}_n(E)} \dim_{\mathcal{A}} M^{\Gamma_{\tilde{x}}} \\ &= \sum_{\varepsilon \in \mathcal{P}_n} \sum_{e \in \Gamma_\varepsilon \backslash X_\varepsilon^+} \dim_{\mathcal{A}} M^{\Gamma_{(\varepsilon,e)}} \\ &= \sum_{\varepsilon \in \mathcal{P}_n} \ell(\varepsilon) \dim_{\mathcal{A}} M^{\Gamma_\varepsilon^0}. \end{aligned}$$

□

Definition 3.2. Define two finiteness conditions on Γ, X, and ρ:

1. $\dim_{\mathcal{A}} H_\rho(EX) < \infty$.
2. For all n, $\dim_{\mathcal{A}} H_\rho(\mathcal{E}_n(E)) < \infty$.

Note that $\dim_{\mathcal{A}} H_\rho(EX) < \infty$ implies $\dim_{\mathcal{A}} H_\rho(VX) < \infty$, because $H_\rho(VX)$ can be identified with the subspace of $H_\rho(EX)$ of functions which are constant on edges that share initial vertices.

Remark 1. Finiteness condition 2 is equivalent to the condition:

2′. For all n,

$$\sum_{\substack{\varepsilon\in\mathcal{P};\\ \ell(\varepsilon)=n}} \dim_{\mathcal{A}} M^{\Gamma_\varepsilon^0} < \infty \tag{3.10}$$

Proof. $\mathcal{E}_n(E)$ is naturally a disjoint union over $k|n$ of subsets $\{(\varepsilon,e)\mid \ell(\varepsilon)=k\}$. Then $H_\rho(\mathcal{E}_n(E))$ splits as a finite direct sum over $k|n$, and the dimension of the k^{th} summand is exactly k times the left hand side of (3.10). □

Both finiteness conditions hold when Γ is a uniform tree lattice and $\dim_{\mathcal{A}} M < \infty$. This follows easily from the fact that $\Gamma\backslash EX$ and $\Gamma\backslash\mathcal{E}_n(E)$ are finite sets. If Γ is a non-uniform X-lattice, the conditions will hold if M and ρ are chosen appropriately. See Section 5.2 for more details.

Definition 3.3. Suppose X, Γ, and ρ satisfy the two finiteness conditions of Definition 3.2. Define σ_ε as in (3.3), and the zeta function associated to X, Γ, and ρ as

$$Z_\rho(u) = \prod_{\varepsilon\in\mathcal{P}} \mathrm{Det}_{\mathcal{A}}(I-\rho(\sigma_\varepsilon)u^{\ell(\varepsilon)}) \quad \in \mathbb{C}[[u]] \tag{3.11}$$

In contrast with the zeta function in [2], there may very well be infinitely many $\varepsilon\in\mathcal{P}$ with the same length. This means one must check that the coefficient of u^n is finite for each n, in order that $Z_\rho(u)\in\mathbb{C}[[u]]$. This question is answered positively in the proof of Theorem 3.4.

3.4. **Proof of the Bass-Hashimoto formula.** In this section, we prove Theorem 0.1. Formally, the proof is similar to the arguments of [2]. However there are a number of convergence questions which must be addressed.

Theorem 3.4. *Suppose X, Γ, and ρ satisfy the two finiteness conditions:*

1. $\dim_{\mathcal{A}} H_\rho(EX) < \infty$.
2. *For all n,* $\dim_{\mathcal{A}} H_\rho(\mathcal{E}_n(E)) < \infty$.

Then $Z_\rho(u)$ is well defined and

$$Z_\rho(u) = \mathrm{Det}_{\mathcal{A}}(I-T_\rho u) \quad \in \mathbb{C}[[u]]. \tag{3.12}$$

In addition, the power series $Z_\rho(u)$ converges for $u\in\mathbb{C}$, $|u|<\|T_\rho\|$.

Proof. Convergence for small u will follow from (3.12) and Proposition 2.7. So, compute formally:

$$\begin{aligned}\operatorname{Log}\mathrm{Det}_{\mathcal{A}}(I-T_\rho u) &= \mathrm{Tr}_{\mathcal{A}}\operatorname{Log}(I-T_\rho u)\\ &= -\sum_{n=1}^{\infty}\frac{u^n}{n} Tr_{\mathcal{A}} T_\rho^n\\ &= -\sum_{n=1}^{\infty}\frac{u^n}{n}\left(\sum_{\varepsilon\in\mathcal{P}_n(\Gamma)} \ell(\varepsilon)\,\mathrm{Tr}_{\mathcal{A}}\,\rho(\sigma_\varepsilon^{n/\ell(\varepsilon)})\right),\end{aligned} \tag{3.13}$$

from Lemma 3.5. The coefficient of u^n converges absolutely for each n.

Rearranging terms and substituting $n = m\ell(\varepsilon)$, (3.13) becomes

$$\begin{aligned}
\operatorname{Log}\operatorname{Det}_{\mathcal{A}}(I - T_\rho u) &= -\sum_{\varepsilon\in\mathcal{P}(\Gamma)}\sum_{m=1}^{\infty}\frac{u^{m\ell(\varepsilon)}}{m}\operatorname{Tr}_{\mathcal{A}}\rho(\sigma_\varepsilon^m) \\
&= \sum_{\varepsilon\in\mathcal{P}(\Gamma)}\operatorname{Tr}_{\mathcal{A}}\left(-\sum_{m=1}^{\infty}\rho(\sigma_\varepsilon^m)\frac{u^{m\ell(\varepsilon)}}{m}\right) \\
&= \sum_{\varepsilon\in\mathcal{P}(\Gamma)}\operatorname{Tr}_{\mathcal{A}}\operatorname{Log}\left(I - \rho(\sigma_\varepsilon)u^{\ell(\varepsilon)}\right) \\
&= \sum_{\varepsilon\in\mathcal{P}(\Gamma)}\operatorname{Log}\operatorname{Det}_{\mathcal{A}}\left(I - \rho(\sigma_\varepsilon)u^{\ell(\varepsilon)}\right) \\
&= \operatorname{Log} Z_\rho(u).
\end{aligned}$$

□

Lemma 3.5. *For all $n = 1, 2, \ldots,$*

$$\operatorname{Tr}_{\mathcal{A}} T_\rho^n = \sum_{\varepsilon\in\mathcal{P}_n(\Gamma)}\ell(\varepsilon)\operatorname{Tr}_{\mathcal{A}}\rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}). \tag{3.14}$$

The sum converges absolutely.

Proof. First we prove that the right hand side of (3.14) is absolutely convergent. Notice that $\rho(\sigma_\varepsilon^{n/\ell(\varepsilon)})$ is a projection to $M^{\Gamma_\varepsilon^0}$ followed by a unitary operator, hence by (2.1)

$$|\operatorname{Tr}_{\mathcal{A}}\rho(\sigma_\varepsilon^{n/\ell(\varepsilon)})| \le \dim_{\mathcal{A}} M^{\Gamma_\varepsilon^0}.$$

Using Lemma 3.1

$$\sum_{\varepsilon\in\mathcal{P}_n(\Gamma)}\ell(\varepsilon)|\operatorname{Tr}_{\mathcal{A}}\rho(\sigma_\varepsilon^{n/\ell(\varepsilon)})| \le \dim_{\mathcal{A}} H_\rho(\mathcal{E}_n(E))$$

which is finite by assumption.

As in Section 2.5, fix a lift $\widetilde{e} \in EX$ of each edge $e \in EB$. We then have an $\mathcal{A}$-equivariant isometric embedding $H_\rho(EX) \hookrightarrow \hat{\bigoplus}_{e\in EB} M$. To compute the trace of T_ρ^n, we calculate the trace of T_ρ^n acting on the e^{th} summand for each $e \in EB$. More precisely, for $e \in EB$ let $(T_\rho^n)_e$ denote the composition

$$M \to M^{\Gamma_{\widetilde{e}}} \hookrightarrow H_\rho(EX) \xrightarrow{T_\rho^n} H_\rho(EX) \to M. \tag{3.15}$$

The first map in (3.15) is the projection given by averaging over $\Gamma_{\widetilde{e}}$. The second map is $\psi \to f_\psi$ described in (2.4). The final map to M is evaluation at $\widetilde{e}$. Thus

$$(T_\rho^n)_e = \left(T_\rho^n f_{\rho(\delta_{\widetilde{e}})\phi}\right)(\widetilde{e}),$$

where

$$\delta_{\widetilde{e}} = \frac{1}{|\Gamma_{\widetilde{e}}|}\sum_{g\in\Gamma_{\widetilde{e}}} g.$$

Using the definition of T_ρ, for any $\phi \in M$ we have

$$\begin{aligned}
(T_\rho^n)_e\phi &= \sum_{(\widetilde{e},e_2,\dots,e_n)\text{ red.}} f_{\rho(\delta_{\widetilde{e}})\phi}(e_n) \\
&= \sum_{(\widetilde{e},e_2,\dots,e_n)\text{ red.}} \begin{cases} f_{\rho(\delta_{\widetilde{e}})\phi}(\gamma\widetilde{e}) & \text{if } e_n = \gamma\widetilde{e} \text{ for some } \gamma\in\Gamma, \\ 0 & \text{otherwise.}\end{cases} \\
&= \sum_{(\widetilde{e},e_2,\dots,e_n)\text{ red.}} \begin{cases} \rho(\gamma)\rho(\delta_{\widetilde{e}})\phi & \text{if } e_n = \gamma\widetilde{e} \text{ for some } \gamma\in\Gamma, \\ 0 & \text{otherwise.}\end{cases} \\
&= \sum_{(\widetilde{e},e_2,\dots,e_n)\text{ red.}} \begin{cases} \rho(\gamma\delta_{\widetilde{e}})\phi & \text{if } e_n = \gamma\widetilde{e} \text{ for some } \gamma\in\Gamma, \\ 0 & \text{otherwise.}\end{cases} \\
&= \sum_{(\widetilde{e},e_2,\dots,e_n)\text{ red.}} \rho\left(\frac{1}{|\Gamma_{\widetilde{e}}|}\sum_{\gamma\in\Gamma,\gamma\widetilde{e}=e_n}\gamma\right)\phi \\
&= \sum_{\substack{\gamma\text{ hyp.};\ell(\gamma)=n;\\ \widetilde{e}\in X_\gamma^+}} \frac{1}{|\Gamma_{\widetilde{e}}|}\rho(\gamma)\phi.
\end{aligned}$$

We can now take the trace, to obtain

$$\operatorname{Tr}_{\mathcal{A}}(T_\rho^n)_e = \sum_{\substack{\gamma\text{ hyp.};\ell(\gamma)=n;\\ \widetilde{e}\in X_\gamma^+}} \frac{1}{|\Gamma_{\widetilde{e}}|}\operatorname{Tr}_{\mathcal{A}}\rho(\gamma). \tag{3.16}$$

Bass [2, II.2.4(5)] shows that all hyperbolic elements associated to an end have the same axis. We can therefore change (3.16) to a sum over ends.

We have

$$\begin{aligned}
\operatorname{Tr}_{\mathcal{A}}(T_\rho^n)_e &= \sum_{\substack{\varepsilon\in\mathcal{E}_n;\\ \widetilde{e}\in X_\varepsilon^+}}\sum_{\substack{\gamma\in\Gamma_\varepsilon;\\ \tau(\gamma)=n}} \frac{1}{|\Gamma_{\widetilde{e}}|}\operatorname{Tr}_{\mathcal{A}}\rho(\gamma) \\
&= \sum_{\substack{\varepsilon\in\mathcal{E}_n;\\ \widetilde{e}\in X_\varepsilon^+}} \frac{|\Gamma_\varepsilon^0|}{|\Gamma_{\widetilde{e}}|}\operatorname{Tr}_{\mathcal{A}}\rho\left(\frac{1}{|\Gamma_\varepsilon^0|}\sum_{\substack{\gamma\in\Gamma_\varepsilon;\\ \tau(\gamma)=n}}\gamma\right) \qquad (3.17)\\
&= \sum_{\substack{\varepsilon\in\mathcal{E}_n;\\ \widetilde{e}\in X_\varepsilon^+}} \left[\Gamma_{\widetilde{e}}:\Gamma_\varepsilon^0\right]^{-1}\operatorname{Tr}_{\mathcal{A}}\rho\left(\sigma_\varepsilon^{n/\ell(\varepsilon)}\right).
\end{aligned}$$

The trace of T_ρ^n is the sum over EB of $\operatorname{Tr}_{\mathcal{A}}(T_\rho^n)_e$. That is, we have expressed the trace as a sum over orbit representatives of edges and then a sum over all ends. The key step of this computation is to switch to a sum over orbit representatives of ends and then a sum over all edges.

Note that each $\Gamma_{\widetilde{e}}$ orbit of $\varepsilon \in \mathcal{E}_n(\widetilde{e})$ has $\left[\Gamma_{\widetilde{e}} : \Gamma^0_\varepsilon\right]$ elements. Also, the final summand in (3.17) is Γ-invariant as a function of ε. Thus,

$$\begin{aligned}
\operatorname{Tr}_{\mathcal{A}} T^n_\rho &= \sum_{e\in EB} \sum_{\varepsilon\in\mathcal{E}_n(\widetilde{e})} \left[\Gamma_{\widetilde{e}} : \Gamma^0_\varepsilon\right]^{-1} \operatorname{Tr}_{\mathcal{A}} \rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}) \\
&= \sum_{e\in EB} \sum_{\varepsilon\in\Gamma_{\widetilde{e}}\backslash\mathcal{E}_n(\widetilde{e})} \operatorname{Tr}_{\mathcal{A}} \rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}) \\
&= \sum_{(e,\varepsilon)\in\Gamma\backslash\mathcal{E}_n(E)} \operatorname{Tr}_{\mathcal{A}} \rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}) \\
&= \sum_{\varepsilon\in\Gamma\backslash\mathcal{E}_n} \sum_{e\in\Gamma_\varepsilon\backslash X_\varepsilon} \operatorname{Tr}_{\mathcal{A}} \rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}) \\
&= \sum_{\varepsilon\in\mathcal{P}_n} \ell(\varepsilon) \operatorname{Tr}_{\mathcal{A}} \rho(\sigma_\varepsilon^{n/\ell(\varepsilon)}).
\end{aligned} \tag{3.18}$$

The interchange of summations is allowed because the last sum converges absolutely. □

3.5. **The operator $\Delta_\rho(u)$.** We prove Theorem 0.2 from the introduction. Continue with the notation of Section 3.3, and suppose that $\dim_{\mathcal{A}} H_\rho(EX) < \infty$ (and therefore $\dim_{\mathcal{A}} H_\rho(VX) < \infty$).

Theorem 3.6. *Put* $\chi_\rho(X) = \dim_{\mathcal{A}} H_\rho(VX) - \frac{1}{2}\dim_{\mathcal{A}} H_\rho(EX)$. *Then*

$$\operatorname{Det}_{\mathcal{A}}(I - T_\rho u) = (1-u^2)^{-\chi_\rho(X)} \operatorname{Det}_{\mathcal{A}}(I - \delta_\rho u + Q_\rho u^2) \tag{3.19}$$

where exponentiation is defined using the principal branch of the logarithm.

Proof. The proof is very similar to that of [2, Theorem II.1.5].

Define the operators:

$$\begin{aligned}
\partial_0, \partial_1 &: H_\rho(VX) \to H_\rho(EX) \\
\partial_i f(e) &= f(\partial_i e); \quad i = 0,1 \\
\sigma_0 &: H_\rho(EX) \to H_\rho(VX) \\
\sigma_0 g(x) &= \sum_{\partial_0 e = x} g(e) \\
J &: H_\rho(EX) \to H_\rho(EX) \\
Jg(e) &= g(\bar{e})
\end{aligned}$$

Notice that $\sigma_0\partial_0 = I_V + Q_\rho$ and $\sigma_0\partial_1 = \delta_\rho$.

Let $H = H_\rho(VX) \bigoplus H_\rho(EX)$ and define the following two operators on H:

$$L = \begin{bmatrix} (1-u^2)I_V & 0 \\ u\partial_0 - \partial_1 & I_E \end{bmatrix}, \qquad M = \begin{bmatrix} I_V & u\sigma_0 \\ u\partial_1 - \partial_0 & (1-u^2)I_E \end{bmatrix}$$

A simple computation shows that

$$ML = \begin{bmatrix} \Delta_\rho(u) & \sigma_0 u \\ 0 & (1-u^2)I_E \end{bmatrix}$$

and

$$LM = \begin{bmatrix} (1-u^2)I_V & (1-u^2)\sigma_0 u \\ 0 & (I_E - T_\rho u)(I_E - Ju) \end{bmatrix}.$$

M is invertible ([2, II-1.6]) so $\mathrm{Det}_{\mathcal{A}}(LM) = \mathrm{Det}_{\mathcal{A}}(M(LM)M^{-1}) = \mathrm{Det}_{\mathcal{A}}(ML)$ by Proposition 2.6.2.

Using Proposition 2.6, compute

$$\begin{aligned}\mathrm{Det}_{A}(ML) &= \mathrm{Det}_{\mathcal{A}}(\Delta_\rho(u))\,\mathrm{Det}_{\mathcal{A}}(1-u^2)I_E \\ &= (1-u^2)^{\dim_{\mathcal{A}} H_\rho(EX)}\,\mathrm{Det}_{\mathcal{A}}(\Delta_\rho(u))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Det}_{\mathcal{A}}(LM) &= \mathrm{Det}_{\mathcal{A}}(I-T_\rho u)(I_E-Ju)\,\mathrm{Det}_{\mathcal{A}}((1-u^2)I_V) \\ &= \mathrm{Det}_{\mathcal{A}}(I-T_\rho u)\,\mathrm{Det}_{\mathcal{A}}(I_E-Ju)\,\mathrm{Det}_{\mathcal{A}}((1-u^2)I_V) \\ &= (1-u^2)^{\dim_{\mathcal{A}} H_\rho(VX)}\,\mathrm{Det}_{\mathcal{A}}(I-T_\rho u)\,\mathrm{Det}_{\mathcal{A}}(I_E-Ju) \\ &= (1-u^2)^{\dim_{\mathcal{A}} H_\rho(VX)+\dim_{\mathcal{A}} H_\rho(EX)/2}\,\mathrm{Det}_{\mathcal{A}}(I-T_\rho u).\end{aligned}$$

The last equality is the computation $\mathrm{Det}_{\mathcal{A}}(I_E-Ju) = \frac{1}{2}\dim_{\mathcal{A}} H_\rho(EX)$ in [2, II-1.6]. The theorem follows easily.

□

4. Coset Representations

The coset representation of an infinite group on the Hilbert space of L^2 functions on a quotient group is the motivating example for this paper. In this section, the theorems of Section 3 are interpreted for coset representations.

Suppose there is an exact sequence $1 \to \Lambda \to \Gamma \to \pi \to 1$. Take $\mathcal{A} = \mathcal{N}(\pi)$ (see Example 2.3), $M = l^2(\pi)$, and ρ the coset representation of Γ on M:

$$(\rho(\gamma)\phi)(x) = \phi(\gamma^{-1}x)$$

for $\gamma \in \Gamma$, $x \in \pi, \phi \in l^2(\pi)$. This situation will hold for the entire section.

4.1. **The spaces H_ρ.** Suppose Γ acts on a countable discrete set X, $B = \Gamma\backslash X$, and set $Y = \Lambda\backslash X$. We have the natural action of π on Y and $\pi\backslash Y = B$. For $y \in Y$, let $\pi_y = \{\alpha \in \pi \mid \alpha y = y\}$.

Proposition 4.1. *The von Neumann dimension of $H_\rho(X)$ is given by*

$$\dim_{\mathcal{N}(\pi)} H_\rho(X) = \sum_{x\in B} \frac{1}{|\pi_{\hat{x}}|}$$

where $\hat{x} \in Y$ is any lift of x.

Proof. Fix $x \in B$, with lift $\widetilde{x} \in X$ and $\hat{x} \in Y$. From (2.3), we need to compute $\dim_\pi l^2(\pi)^{\Gamma_{\widetilde{x}}}$ for each $x \in B$. First notice that $\Gamma_{\widetilde{x}}/(\Gamma_{\widetilde{x}} \cap \Lambda) \cong \pi_{\hat{x}}$.

The action of Γ on $l^2(\pi)$ factors through π, and so

$$\begin{aligned}l^2(\pi)^{\Gamma_{\widetilde{x}}} &= l^2(\pi)^{\pi_{\hat{x}}} \\ &= \{f \in l^2(\pi) \mid f \text{ is constant on cosets of } \pi_{\hat{x}}\}.\end{aligned}$$

Orthogonal projection P onto $l^2(\pi)^{\Gamma_{\widetilde{x}}}$ is therefore given by averaging over $\pi_{\hat{x}}$, and so $\langle P(\delta_{1_\pi}), \delta_{1_\pi}\rangle = |\pi_{\hat{x}}|^{-1}$. □

Remark 2. Since Λ acts trivially on $l^2(\pi)$, functions in $H_\rho(X)$ are periodic in the Λ direction, so one can think of them as functions on the quotient Y. In fact, when

Λ acts freely on X it is not hard to check that $H_\rho(X)$ and $l^2(Y)$ are isometric as $\mathcal{N}(\pi)$ modules by the map $\Upsilon : H_\rho(X) \to l^2(Y)$, where

$$(\Upsilon f)(x) = |\pi_{\hat{x}}|^{\frac{1}{2}} f(\widetilde{x})(1_\pi).$$

4.2. **The zeta function.** Now suppose that X is a tree and Γ is an X-lattice. Put $Y = \Lambda\backslash X$. As before, we let $\mathcal{E}(\Gamma)$ denote the set of hyperbolic ends of Γ, and $\mathcal{P}(\Gamma) = \Gamma\backslash\mathcal{E}(\Gamma)$. We also denote the hyperbolic ends of Λ by $\mathcal{E}(\Lambda)$ and set $\mathcal{P}(\Lambda) = \Lambda\backslash\mathcal{E}(\Lambda)$. There is a natural action of π on $\mathcal{P}(\Lambda)$ and we have $\pi\backslash\mathcal{P}(\Lambda) = \Gamma\backslash\mathcal{E}(\Lambda)$.

As before, we denote the length of an end $\varepsilon \in \mathcal{E}(\Gamma)$ by $\ell(\varepsilon)$. Since we have an injection $\mathcal{E}(\Lambda) \hookrightarrow \mathcal{E}(\Gamma)$, some ends of Γ are also ends of Λ and therefore have a possibly greater length. To emphasize the difference, we put

$$\ell_\Lambda(\varepsilon) = \min\{\ell(g) \mid g\varepsilon = \varepsilon; g \text{ hyperbolic}; g \in \Lambda\}$$

Theorem 4.2. *When $Z_\rho(u)$ is well defined,*

$$Z_\rho(u) = \prod_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \left(1 - u^{\ell_\Lambda(\varepsilon)}\right)^{\frac{1}{|\pi_\varepsilon|}} \tag{4.1}$$

where $\pi_\varepsilon = \{\alpha \in \pi \mid \alpha \cdot \varepsilon = \varepsilon \in \mathcal{P}(\Lambda)\}$.

Proof. From (3.11) we have

$$Z_\rho(u) = \prod_{\varepsilon\in\mathcal{P}(\Gamma)} \mathrm{Det}_\pi(I - \rho(\sigma_\varepsilon)u^{\ell(\varepsilon)})$$

so that

$$\begin{aligned} \operatorname{Log} Z_\rho(u) &= \sum_{\varepsilon\in\mathcal{P}(\Gamma)} \mathrm{Tr}_\pi \operatorname{Log}(I - \rho(\sigma_\varepsilon)u^{\ell(\varepsilon)}) \\ &= -\sum_{\varepsilon\in\mathcal{P}(\Gamma)} \sum_{n=1}^{\infty} \frac{u^{n\ell(\varepsilon)}}{n} \mathrm{Tr}_\pi \rho(\sigma_\varepsilon^n). \end{aligned} \tag{4.2}$$

Now fix $\varepsilon \in \mathcal{P}(\Gamma)$. Fix $n \in \mathbb{N}$. Choose any $\gamma \in \Gamma_\varepsilon$ with $\ell(\gamma) = n\ell(\varepsilon)$. We have

$$\begin{aligned} \mathrm{Tr}_\pi \rho(\sigma_\varepsilon^n) &= \mathrm{Tr}_\pi \rho\left(\frac{1}{|\Gamma_\varepsilon^0|}\sum_{g\in\Gamma_\varepsilon^0} \gamma g\right) \\ &= \frac{1}{|\Gamma_\varepsilon^0|}\sum_{g\in\Gamma_\varepsilon^0} \begin{cases} 1 & \text{if } \gamma g \in \Lambda; \\ 0 & \text{if } \gamma g \notin \Lambda. \end{cases} \end{aligned} \tag{4.3}$$

If $\varepsilon \in \mathcal{P}(\Gamma) - \pi\backslash\mathcal{P}(\Lambda)$, no hyperbolic element of Γ_ε lies in Λ. So, for such ε, $\mathrm{Tr}_\pi \rho(\sigma_\varepsilon^n) = 0$.

Define $\Lambda_\varepsilon^0 = \Gamma_\varepsilon^0 \cap \Lambda$. Then for $\varepsilon \in \pi\backslash\mathcal{P}(\Lambda)$,

$$\mathrm{Tr}_\pi \rho(\sigma_\varepsilon^n) = \begin{cases} \frac{|\Lambda_\varepsilon^0|}{|\Gamma_\varepsilon^0|} & \text{if } \ell_\Lambda(\varepsilon) \text{ divides } n\ell(\varepsilon); \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Now, putting $k = n\ell(\varepsilon)/\ell_\Lambda(\varepsilon)$, (4.2) becomes

$$\begin{aligned}\operatorname{Log} Z_\rho(u) &= -\sum_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \sum_{k=1}^{\infty} \frac{\ell(\varepsilon)u^{k\ell_\Lambda(\varepsilon)}}{k\ell_\Lambda(\varepsilon)} \left(\frac{|\Lambda_\varepsilon^0|}{|\Gamma_\varepsilon^0|}\right) \\ &= -\sum_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \frac{1}{|\pi_\varepsilon|} \sum_{k=1}^{\infty} \frac{u^{k\ell_\Lambda(\varepsilon)}}{k} \quad \text{(using Lemma 4.3)} \\ &= \sum_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \frac{1}{|\pi_\varepsilon|} \operatorname{Log}\left(1 - u^{\ell_\Lambda(\varepsilon)}\right) \\ &= \operatorname{Log} \prod_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \left(1 - u^{\ell_\Lambda(\varepsilon)}\right)^{\frac{1}{|\pi_\varepsilon|}}\end{aligned} \tag{4.5}$$

which proves the theorem. ☐

Lemma 4.3. *With the notation of Theorem 4.2,*

$$|\pi_\varepsilon| = \frac{|\ell_\Lambda(\varepsilon)||\Gamma_\varepsilon^0|}{|\ell(\varepsilon)||\Lambda_\varepsilon^0|}.$$

Proof. Define $\pi_\varepsilon^0 = \Gamma_\varepsilon^0/\Lambda_\varepsilon^0$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Lambda_\varepsilon^0 & \longrightarrow & \Gamma_\varepsilon^0 & \longrightarrow & \pi_\varepsilon^0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Lambda_\varepsilon & \longrightarrow & \Gamma_\varepsilon & \longrightarrow & \pi_\varepsilon & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \ell_\Lambda(\varepsilon)\mathbb{Z} & \longrightarrow & \ell(\varepsilon)\mathbb{Z} & \longrightarrow & \mathbb{Z}_{\ell_\Lambda(\varepsilon)/\ell(\varepsilon)} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

It is not difficult to check that all rows and columns are exact, and the lemma follows. ☐

4.3. **Infinite graphs.** Suppose that Λ acts freely on X. Then the finiteness conditions (Definition 3.2) needed to define the zeta function and prove Theorem 0.1 are always satisfied. This is the content of the following theorem.

Theorem 4.4. *Suppose* Γ *is an* X*-lattice where* X *is a tree of bounded degree. Let* $1 \to \Lambda \to \Gamma \to \pi \to 1$*, and let* ρ *be the coset representation of* Γ *on* $l^2(\pi)$*. If* Λ *is free, then*

1. $\dim_{\mathcal{N}(\pi)} H_\rho(EX) < \infty$.
2. *For all* n, $\dim_{\mathcal{N}(\pi)} H_\rho(\mathcal{E}_n(E)) < \infty$.

Proof. 1. For any $e \in EB$, let $\widetilde{e}$ be a lift of e to EX, and let $\hat{e}$ be the edge in EY covered by $\widetilde{e}$. We have $\Gamma_{\widetilde{e}} = \Gamma_{\widetilde{e}}/(\Gamma_{\widetilde{e}} \cap \Lambda) \cong \pi_{\hat{e}}$ because Λ is free and $\Gamma_{\widetilde{e}}$ is torsion. Then by Proposition 4.1, $\dim_{\mathcal{N}(\pi)} H_\rho(EX)$ is the edge-volume of $\Gamma\backslash X$ (Definition 1.2), which is finite.

2. By combining Lemma 3.1 with the computation in Proposition 4.1,

$$\begin{aligned}\dim_{\mathcal{N}(\pi)} H_\rho(\mathcal{E}_n(E)) &= \sum_{\varepsilon\in\mathcal{P}_n} \ell(\varepsilon) \dim_{\mathcal{N}(\pi)} l^2(\pi)^{\Gamma^0_\varepsilon} \\ &= \sum_{\varepsilon\in\mathcal{P}_n} \ell(\varepsilon) \frac{1}{|\Gamma^0_\varepsilon|} \\ &\leq n \sum_{\varepsilon\in\mathcal{P}_n} \frac{1}{|\Gamma^0_\varepsilon|}. \qquad (4.6)\end{aligned}$$

Let d be the bound on the degree of vertices of X. We claim that $|\Gamma^0_\varepsilon| \geq d^{-n}|\Gamma_{\widetilde{e}}|$, where $\widetilde{e} \in X_\varepsilon$. To see the claim, let $\varepsilon = \varepsilon(g)$. Then the orbit of $g\widetilde{e}$ under $\Gamma_{\widetilde{e}}$ has at most $d^{\ell(\varepsilon)} \leq d^n$ elements. The group Γ^0_ε is the stabilizer in $\Gamma_{\widetilde{e}}$ of $g\widetilde{e}$, and this proves the claim.

Choose a lift $\widetilde{e}$ for each edge $e \in EB$. Each class $\varepsilon \in \mathcal{P}_n$ has a representative which contains at least one $\widetilde{e}$. On the other hand, each edge $\widetilde{e} \in EX$ is contained in only finitely many $\varepsilon \in \mathcal{P}_n$ (a loose bound is d^n). Then continuing (4.6),

$$n \sum_{\varepsilon\in\mathcal{P}_n} \ell(\varepsilon)\frac{1}{|\Gamma^0_\varepsilon|} \leq nd^{2n} \sum_{e\in EB} \frac{1}{|\Gamma_{\widetilde{e}}|}$$

which is finite.

□

Combining Theorem 3.4, Theorem 3.6, and Theorem 4.2, one arrives at the formula

$$Z_\rho(u) = \prod_{\varepsilon\in\pi\backslash\mathcal{P}(\Lambda)} \left(1 - u^{\ell_\Lambda(\varepsilon)}\right)^{\frac{1}{|\pi_\varepsilon|}} = (1-u^2)^{-\chi_\rho(X)} \operatorname{Det}_{\mathcal{A}}(I - \delta_\rho u + Q_\rho u^2). \quad (4.7)$$

The remainder of this section consists of interpretation of (4.7).

When Λ is free, $\mathcal{P}(\Lambda)$ has a nice geometric interpretation as primitive, tail-less, backtrack-less loops in Y (or equivalently free homotopy classes). The length of the loop associated to ε is $\ell_\Lambda(\varepsilon)$. π acts on these loops by translation, so the product in (4.7) is over translation classes of loops. The group π_ε is the subgroup of π which fixes the loop associated to ε.

Since the preceding discussion reformulates the definition of $Z_\rho(u)$ completely in terms of Y and π, it makes sense to refer to the *π-zeta function of the infinite graph Y*,

$$Z_\pi(Y, u) = Z_\rho(u) = \prod_{\gamma\in\pi\backslash P} \left(1 - u^{\ell(\gamma)}\right)^{\frac{1}{|\pi_\gamma|}}. \quad (4.8)$$

Here $P = \mathcal{P}(\Lambda)$.

It remains to interpret the $\Delta(u)$ side of formula (4.7). There are isomorphisms of $H_\rho(EX)$ with $l^2(EY)$ and of $H_\rho(VX)$ with $l^2(VY)$ as described in Remark 2. The operator δ_ρ becomes the usual adjacency operator on Y and Q_ρ is multiplication at $x \in VY$ by $deg(x) - 1$. Then $\Delta(u)$ is just a weighted version of the classical Laplace operator on l^2 0-chains of Y.

Finally, the Euler characteristic $\chi_\rho(X)$ is the L^2 Euler characteristic $\chi^{(2)}(Y)$, as in [4], and is easily computable. If π acts freely on Y, $\chi^{(2)}(Y) = \chi(B)$. Otherwise, using Proposition 4.1,

$$\chi^{(2)}(Y) = \sum_{x\in VB} \frac{1}{|\pi_x|} - \frac{1}{2}\sum_{e\in EB}\frac{1}{|\pi_e|}.$$

This expression is also studied in [4] where it is referred to as the virtual Euler characteristic of B.

Under the identifications in this section, (4.7) becomes

$$Z_\pi(Y,u) = (1-u^2)^{-\chi^{(2)}(Y)}\operatorname{Det}_\pi(\Delta(u)).$$

This completes the proof of Theorem 0.3.

5. Examples

This section is concerned with the situation of Section 4. That is, there is an exact sequence $1 \to \Lambda \to \Gamma \to \pi \to 1$, and $Y = \Lambda\backslash X$. We take ρ to be the coset representation of Γ on $l^2(\pi)$.

5.1. **The infinite grid.** We examine Theorem 0.3 for a concrete example. Let $\pi = \mathbb{Z}\times\mathbb{Z} = \langle a\rangle \times \langle b\rangle$. Let Y be the Cayley graph of π with generators $(a,0)$ and $(0,b)$. Y is an infinite grid. The formula gives a method for computing the number $N(L)$ of translation classes of tail-less, backtrack-less primitive loops in Y of length L. These are exactly the number of possible shapes of closed paths on a grid, as should be made clear by the picture to follow.

In this example, π acts freely on all loops (this is true whenever π is torsion-free), so

$$Z_\pi(Y,u) = \prod_{\gamma\in\pi\backslash P}(1-u^{\ell(\gamma)}) = \prod_{l=1}^{\infty}(1-u^l)^{N(l)}.$$

Notice that $N(l)=0$ for l odd (Y is bipartite). Then

$$\begin{aligned} -\operatorname{Log} Z_\pi(Y,u) &= -\sum_{l=1}^{\infty} N(l)\operatorname{Log}(1-u^l) \\ &= -\sum_{m=1}^{\infty}\sum_{l|m} N(l)\frac{l}{m}u^m \\ &= -\sum_{M=1}^{\infty}\sum_{L|M} N(2L)\frac{L}{M}u^{2M}. \end{aligned} \tag{5.1}$$

Computing coefficients of this power series will then compute (inductively) the numbers $N(2L)$. To achieve this, examine the $\Delta(u)$ side of the formula in Theorem 0.3. We have

$$\begin{aligned} -\operatorname{Log} Z_\pi(Y,u) &= \operatorname{Tr}_\pi(-\operatorname{Log}(\Delta(u))) + \sum_{M=1}^{\infty}\frac{1}{M}u^{2M} \\ &= \sum_{n=1}^{\infty}\operatorname{Tr}_\pi(1-\Delta(u))^n + \sum_{M=1}^{\infty}\frac{1}{M}u^{2M} \end{aligned} \tag{5.2}$$

The weighted Laplacian $\Delta(u)$ acts on $l^2(VY)\cong l^2(\pi)$. We write it as a 1×1 matrix with coefficient in $\mathbb{Z}[\pi]$, acting on $l^2(\pi)$ by right multiplication:

$$\Delta(u) = (1-(a+a^{-1}+b+b^{-1})u + 3u^2).$$

Computing $\mathrm{Tr}_\pi(1-\Delta(u))^n$ is an exercise in combinatorics. One finds that

$$-\operatorname{Log} Z_\pi(Y,u) = \sum_{M=1}^{\infty}\left[\frac{1}{M}+\sum_{d=0}^{M}\frac{(-3)^{M-d}}{M+d}\binom{M+d}{M-d}\binom{2d}{d}^2\right]u^{2M}.$$

The first few values of $N(2L)$ are $0,2,4,26,152,1004,\ldots$. The 26 types of loops of length 8 are pictured below (each has two possible orientations).

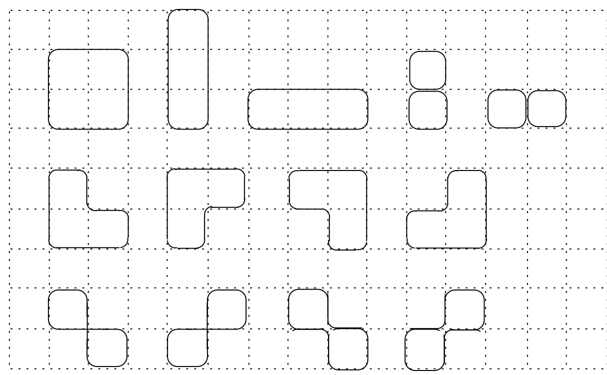

FIGURE 1. Loops of Length 8

5.2. **Non-uniform tree lattices.** If Γ is a uniform X-lattice then any Hilbert representation of Γ on a space of finite von Neumann dimension satisfies the finiteness conditions needed to define the corresponding zeta function. In particular, one can let $\mathcal{A}=\mathbb{C}$ and choose any representation of Γ on a finite dimensional vector space. However, except for the trivial representation there is no interesting canonical choice of representation.

When Γ is a non-uniform lattice the trivial representation of Γ does not satisfy the finiteness condition—$H_\rho(EX)$ is not finite dimensional. However in this case if one can find a free normal subgroup $\Lambda \triangleleft \Gamma$ then the corresponding coset representation yields a well-defined zeta function, by Theorem 4.4.

There are other possible representations for non-uniform lattices. The following example is a coset representation with a well-defined zeta function, where Λ is not free.

Example 5.1. Suppose Γ acts on a $q+1$-regular tree X with quotient as in the diagram:

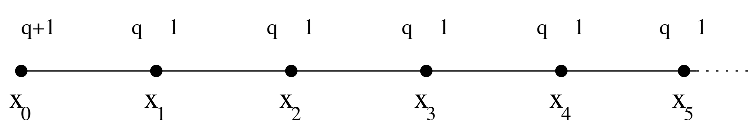

FIGURE 2. The quotient $\Gamma\backslash X$

To understand the figure, notice that it shows only geometric edges. For every oriented edge there is an assigned index which is displayed in the figure at the initial vertex of the edge.

For edge $e = (x_n, x_{n-1})$ the index is q. Then if $\widetilde{x}_n$ is a lift of x_n there are exactly q edges $e_1, \dots, e_q$ covering e with $\partial_0 e_i = \widetilde{x}_n$. The same interpretation holds for other indices.

Any such Γ is always an X-lattice (see [3]).

Choose a lift $(\widetilde{x}_0, \widetilde{x}_1, \dots)$ of the ray $\Gamma\backslash X$ to X. Then $\Gamma = \langle \Gamma_{\widetilde{x}_n}; n = 0, 1, \dots \rangle$, the group generated by the vertex stabilizers. In fact, $\Gamma_{\widetilde{x}_n} < \Gamma_{\widetilde{x}_{n+1}}$ for all $n \geq 1$.

Let $\Lambda = \langle \Gamma_{\widetilde{x}_0}, \Gamma_{\widetilde{x}_1} \rangle_{\text{normal}}$, the normal subgroup generated by $\Gamma_{\widetilde{x}_0}$ and $\Gamma_{\widetilde{x}_1}$ in Γ. Form the exact sequence

$$1 \to \Lambda \to \Gamma \to \pi \to 1$$

where $\pi = \langle \Gamma_{\widetilde{n}_i}; n = 2, 3, \dots \rangle$. It is not hard to show that $\dim_\pi H_\rho(EX) < \infty$ if we assume that $\Gamma_1 \lhd \Gamma_i,\ i \geq 1$. We claim that $\mathcal{P}_n$ is finite for each n and therefore finiteness condition 2 (Definition 3.2) is satisfied. The axis of each hyperbolic element of Γ must pass through some lift of x_0. To count hyperbolic elements up to conjugacy we need only count those whose axes pass through a fixed lift of x_0, and there are only finitely many such axes.

We have shown that the zeta function for Γ and the coset representation ρ of Γ on Λ is well defined.

Using the trivial representation for Γ, the above argument shows that finiteness condition 2 may be satisfied although $\dim_{\mathcal{A}} H_\rho(EX)$ is not finite.

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