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RESIDUAL AMENABILITY AND THE APPROXIMATION OF L^2-INVARIANTS

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ABSTRACT

This thesis generalizes Lück's Theorem to prove that the L^2-Betti numbers of a residually amenable covering space are the limit of the L^2-Betti numbers of a sequence of amenable covering spaces. It is shown that any residually amenable covering space of a finite simplicial complex is of determinant class, and that the L^2-torsion is a homotopy invariant for such spaces. Examples of residually amenable groups are given, including the Baumslag-Solitar groups.

INTRODUCTION

In 1994, Wolfgang Lück [21] proved the beautiful theorem that if X is a finite simplicial complex with residually finite fundamental group, the L^2-Betti numbers of the universal covering of X can be approximated by the ordinary Betti numbers of a sequence of finite coverings of X. In fact, the question of approximation dates back to Kazhdan [18] (see also [15, Pg. 231]) but only an inequality was known. Dodziuk and Mathai [11] have shown a result analogous to Lück's Theorem in the situation where the covering transformation group is amenable. Specifically, they show that the L^2-Betti numbers of an amenable covering $\widetilde{X}$ of X can be approximated by the ordinary Betti numbers of a sequence of Følner subsets of $\widetilde{X}$. This thesis generalizes Lück's Theorem to the case where the cover of X has residually amenable transformation group, a large class of groups that includes the residually finite groups of Lück's Theorem and the amenable groups of Dodziuk and Mathai.

In this thesis, we also consider L^2-torsion. At first, L^2-torsion was defined for L^2-acyclic covering spaces. The L^2-analytic torsion was first studied in [25] and [20], and L^2-Reidemeister-Franz torsion was first studied in [8], see also [23]. Equality of the combinatorial and analytic L^2-torsions was proven in 1996 [5].

In order to define these L^2-torsions, one needs to establish decay near zero of the spectral density function for the L^2-Laplacian. In the case of a residually finite covering, Lück [21] derives an elegant estimate on the spectral density functions for the finite covers, which in the limit gives the necessary decay for the combinatorial L^2-Laplacian. Lück also proves the homotopy invariance of L^2-combinatorial torsion in this case.

In [6], the combinatorial and analytic torsion invariants were defined more generally as volume forms on L^2-cohomology, the decay condition on the spectrum now replaced by a similar condition known as determinant class. The results of [5] extend to show the equality of these more general combinatorial and analytic L^2-torsions.

Dodziuk and Mathai [11] show that coverings with amenable covering group are of determinant class, and Mathai and Rothenberg [26] have recently extended Lück's results to prove the homotopy invariance of L^2-torsion in that case. In this thesis we show that coverings with residually amenable covering group are of determinant class, and that L^2-torsion of such spaces is a homotopy invariant.

On a different note, Farber [12] has recently generalized Lück's Theorem in a new direction, viewing it as a statement about flat bundles rather than finite coverings. In particular, he gives precise conditions for the convergence of L^2-Betti numbers of finite non-regular covers. A reasonable direction for future work would be to try and extend the results of this thesis using his techniques.

We now formulate the main results of this thesis. Let Y be a connected simplicial complex. Suppose that a finitely generated group π acts freely and simplicially on Y so that $X = Y/\pi$ is a finite simplicial complex.

Suppose there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. Form $Y_n = Y/\Gamma_n$, so that $Y_1, Y_2, \ldots$ are a tower of covering spaces of X.

Say that π is residually finite if there exist Γ_n's as above, so that the quotients π/Γ_n are all finite. Then each Y_n is a finite complex, and Lück's Theorem [21] states that

$$b_j^{(2)}(Y : \pi) = \lim_{n\to\infty} \frac{1}{|\Gamma_n|} b_j(Y_n)$$

where $b_j^{(2)}(Y : \pi)$ is the jth L^2-Betti number of Y, and $b_j(Y_n)$ is the ordinary jth Betti number of Y_n.

We generalize Lück's Theorem to the situation where π is residually amenable, meaning there exist Γ_n's as above so that the quotients π/Γ_n are all amenable. The first main result of this thesis is

Theorem 0.1 (Approximation Theorem). *Suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. If π is* *residually amenable, then*

$$b_j^{(2)}(Y:\pi) = \lim_{n\to\infty} b_j^{(2)}(Y_n:\pi/\Gamma_n).$$

The next result gives more evidence for the determinant class conjecture, which states that any regular covering space of a finite simplicial complex is of determinant class. For π residually finite this follows from [21], and it was shown for π amenable in [11].

Theorem 0.2 (Determinant Class Theorem). *Suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. If π is residually amenable, then Y is of determinant class.*

Now we turn to the problem of homotopy invariance of L^2-torsion. The discussion is restricted to closed, compact, odd dimensional manifolds, which necessarily have vanishing Euler characterisic. The L^2-torsions are not in general topological invariants of X unless the Euler characteristic $\chi(X) = 0$.

Let M and N be compact manifolds, $\widetilde{M}$ and $\widetilde{N}$ regular π-covering spaces. As in [26], a homotopy equivalence $f : M \to N$ induces a canonical isomorphism $\widetilde{f}_*^* : \det \overline{H}_{(2)}^*(\widetilde{N}) \to \det \overline{H}_{(2)}^*(\widetilde{M})$ of determinant lines of L^2-cohomology.

Let $\phi_{\widetilde{M}} \in \det \overline{H}_{(2)}^*(\widetilde{M})$ denote the L^2-torsion of M.

Theorem 0.3 (Homotopy Invariance of L^2-Torsion). *Suppose $f : M \to N$ is a homotopy equivalence of closed, compact, odd dimensional manifolds, and $\widetilde{M}$ and $\widetilde{N}$ are regular covering spaces with residually amenable covering group. Then via the above identification of determinant lines of L^2-cohomology,*

$$\phi_{\widetilde{M}} = \phi_{\widetilde{N}} \in \det \overline{H}_{(2)}^*(\widetilde{M}).$$

This provides more evidence for the conjecture in [26] (see also [21]) that L^2-torsion is always a homotopy invariant when the covering spaces in question are of determinant class.

This thesis is organized as follows. The first chapter is a review of L^2-invariants such as the L^2-Betti numbers. Chapter two covers preliminaries on residually amenable groups, and exhibits interesting examples. The third chapter proves the main technical theorem. It essentially states that the L^2-spectra of Laplacians on the Y_n approximate the L^2-spectrum of the Laplacian on Y. Finally, the three main theorems are proved in chapter four.

CHAPTER 1

REVIEW OF L^2-INVARIANTS

This chapter gives the basic definitions of L^2-Betti numbers, L^2-torsion, and determinant class, and surveys some of the important facts and questions about these L^2-invariants.

1.1. Von Neumann Trace

Suppose π is a countable discrete group. Let $l^2(\pi)$ be the Hilbert space of formal sums $\sum_{g\in\pi}\lambda_g \cdot g$ with $\lambda_g \in \mathbb{C}$ and $\sum_{g\in\pi}|\lambda_g|^2 < \infty$. The *von Neumann algebra* of π

$$\mathcal{N}(\pi) = B(l^2(\pi), l^2(\pi))^\pi$$

is the algebra of bounded π-equivariant operators on $l^2(\pi)$, where π is acting on the left on $l^2(\pi)$. It is well known that $\mathcal{N}(\pi)$ is generated (as a von Neumann algebra) by the right multiplication operators R_g, $g \in \pi$. The *von Neumann trace* of $T \in \mathcal{N}(\pi)$ is defined by

$$\mathrm{Tr}_\pi(T) = \langle T(e), e\rangle$$

where e is the identity in π. Equivalently, one can define

$$\mathrm{Tr}_\pi(R_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

and extend linearly.

For a bounded π-equivariant operator $T : \bigoplus_{i=1}^a l^2(\pi) \to \bigoplus_{i=1}^a l^2(\pi)$, we think of T as an $a \times a$ matrix with entries in $\mathcal{N}(\pi)$, and define

$$\mathrm{Tr}_\pi(T) = \sum_{i=1}^a \mathrm{Tr}_\pi(T_{ii}).$$

A *finitely generated Hilbert π-module* H is a Hilbert space with isometric left π action so that there exists an isometric π-equivariant embedding ι of H into $\bigoplus_{i=1}^{a} l^2(\pi)$ for some a. Note that the embedding is not part of the structure. Let $P : \bigoplus_{i=1}^{a} l^2(\pi) \to \bigoplus_{i=1}^{a} l^2(\pi)$ be a projection with image π-equivariantly isometric to H. We write $B(H)$ for the space of bounded, π-equivariant operators on H. For $T \in B(H)$, define

$$\mathrm{Tr}_\pi(T) = \mathrm{Tr}_\pi(\iota \circ T \circ P)$$

and the (real valued) *von Neumann dimension* of H by

$$\mathrm{Dim}_\pi(H) = \mathrm{Tr}_\pi(P).$$

These definitions are independent of the choice of ι and P, and depend only on the isomorphism class of H. The von Neumann dimension is faithful, that is $\mathrm{Dim}_\pi(H) = 0$ implies $H = 0$ [9].

1.2. L^2-Betti Numbers

As in the introduction, suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. Let $C^*_{(2)}(Y)$ denote the L^2-cochain complex of Y. That is,

$$C^j_{(2)}(Y) = \left\{ f \in C^j(Y) \Big| \sum_{j\text{-simplices } \sigma \subset Y} |f(\sigma)|^2 < \infty \right\}.$$

Each $C^j_{(2)}(Y)$ is a Hilbert space with isometric π action, and the coboundaries δ_j are π equivariant bounded operators.

Define the j^{th} *reduced L^2-cohomology* of Y by

$$\overline{H}^j_{(2)}(Y) = \ker(\delta_j)/\overline{\mathrm{im}(\delta_{j-1})}.$$

Let δ_j^* denote the Hilbert space adjoint of δ_j. The combinatorial Laplacian

$$\Delta_j = \delta_j^*\delta_j + \delta_{j-1}\delta_{j-1}^*$$

is a π-equivariant bounded self-adjoint operator on $C^j_{(2)}(Y)$. By the L^2-Hodge decomposition theorem [10], there is an isomorphism of Hilbert $\mathcal{N}(\pi)$-modules,

$$\overline{H}^j_{(2)}(Y) \cong \ker(\Delta_j).$$

If X has a cells in dimension j, we can choose lifts of the cells to Y, and identify $C^j_{(2)}(Y)$ with $\bigoplus_{i=1}^a l^2(\pi)$ (although not canonically). Then $\ker(\Delta_j)$ is embedded in $\bigoplus_{i=1}^a l^2(\pi)$, which makes $\overline{H}^j_{(2)}(Y)$ into a finitely generated Hilbert π-module. We define the L^2-Betti numbers

$$b^j_{(2)}(Y:\pi) = \mathrm{Dim}_\pi(\overline{H}^j_{(2)}(Y)).$$

These numbers are invariants of the π-equivariant homotopy type of Y, [10].

The L^2-Betti numbers were first introduced in 1976 by M. Atiyah [1] in the situation $Y = \widetilde{X}$, the universal cover of X. In that case, one considers the L^2-Betti numbers as homotopy invariants of X, and we write $b^j_{(2)}(X) = b^j_{(2)}(\widetilde{X} : \pi_1(X))$. Atiyah proved an L^2-index theorem from which it follows that

$$\chi(X) = \sum_{j\geq 0} (-1)^j b^j_{(2)}(X).$$

An outstanding question in this field is Atiyah's conjecture that for X a compact manifold, the L^2-Betti numbers are rational, and integral when $\pi_1(X)$ is torsion free. There is also the Singer conjecture that the L^2-Betti numbers of a closed aspherical manifold vanish outside the middle dimension [22].

1.3. Determinant Class and L^2-Torsion

Let H be any finitely generated Hilbert π-module, and let $GL(H)$ denote the group of all invertible operators in $B(H)$. There is a group homomorphism, the *Fuglede-Kadison determinant* $\mathrm{Det}_\pi : GL(H) \to \mathbb{R}^+$, defined in [13], or see [26]. If $T \in GL(H)$ is given by a spectral decomposition

$$T = \int_0^\infty \lambda dE(\lambda) = \int_0^{\|T\|} \lambda dE(\lambda),$$

(e.g. T is positive and self-adjoint) then

$$\log \mathrm{Det}_\pi(T) = \int_0^{\|T\|} \log \lambda dF(\lambda). \tag{1.1}$$

Here, the spectral projections $E(\lambda)$ are in $B(H)$, the spectral density function $F(\lambda) = \mathrm{Tr}_\pi\, E(\lambda)$, and the integral exists as T invertible implies $E(\lambda) = 0$ for small λ. This interpretation of the definition allows one to extend the Fuglede-Kadison determinant to so-called *weakly invertible* operators, which are operators $T \in B(H)$ with spectral decomposition, no kernel, and for which the integral (1.1) exists.

A complex (C, δ) of Hilbert π-modules is *weakly acyclic* if Δ_j is weakly invertible for all j. For a weakly acyclic n-dimensional complex C, the L^2*-Reidemeister-Franz torsion* is defined by

$$\log \phi(C) = \frac{1}{2} \sum_{j=0}^{n} (-1)^{q+1} q \log \mathrm{Det}_\pi(\Delta_j).$$

This is a special case of the Reidemeister-von Neumann torsion introduced by Lück and Rothenberg [23]. The first use of the Fuglede-Kadison determinant in this context was in [7], and then later reworked in [8].

Suppose now $X = Y/\pi$ as above. The combinatorial laplacian Δ_j on Y is a positive self-adjoint operator, and $K = \|\Delta_j\| < \infty$, so by the spectral theorem we may write

$$\Delta_j = \int_0^K \lambda dE_j(\lambda)$$

and then define the spectral density function

$$F_j(\lambda) = \mathrm{Tr}_\pi(E_j(\lambda)).$$

Y is said to be of *determinant class* if for all j, the integral

$$\int_{0+}^K \log \lambda dF_j(\lambda)$$

exists, or equivalently that Δ'_j is weakly invertible, where Δ'_j denotes the restriction of Δ_j to the orthogonal complement of its kernel.

When Y is of determinant class, an L^2-torsion can be defined even if Y is not L^2-acyclic. As long as the Euler characteristic $\chi(X)$ vanishes, the definition of L^2-torsion can be extended to give a topological invariant of Y,

$$\phi_Y \in \det \overline{H}^*_{(2)}(Y)$$

which lies in the determinant line of the L^2-cohomology of Y. The determinant line is the one dimensional vector space of volume forms on $\overline{H}^*_{(2)}(Y)$, defined via the Fuglede-Kadison determinant. For more details, see [6].

There are two fundamental questions concerning these definitions. The first is the determinant class conjecture, which states that every regular covering space of a finite simplicial complex X is of determinant class. Then there is the conjecture that L^2-torsion is in fact a homotopy invariant when the covering spaces in question are of determinant class.

1.4. Analytic L^2-Invariants

If Y is a Riemannian manifold, there are analytic versions of the spectral invariants described in the previous sections. Let $L^2\Lambda^j(Y)$ be the Hilbert space of all square-integrable j-forms on Y. Given a π-equivariant operator T on $L^2\Lambda^j(Y)$ with Schwartz kernel

$$T(x,y) : \Lambda^j T^*_y(Y) \to \Lambda^j T^*_x(Y),$$

define

$$\operatorname{Tr}_\pi T = \int_{\mathcal{F}} \operatorname{tr} T(x,x)dx,$$

where $\mathcal{F}$ is a compact fundamental domain for the action of π on Y. This is independent of the choice of $\mathcal{F}$ [1].

The Laplacian $d^*d + dd^*$ on $L^2\Lambda^j(Y)$ will be denoted by $\widetilde{\Delta}_j$. It is π-equivariant and essentially self-adjoint, so has a spectral family of π-equivariant projections

$\{\widetilde{E}_j(\lambda)|\lambda \in [0,\infty)\}$. Each $E_j(\lambda)$ has Schwartz kernel, and one can then form the analytic spectral density function $\widetilde{F}_j(\lambda) = \mathrm{Tr}_\pi\, \widetilde{E}_j(\lambda)$.

The L^2-Betti numbers were originally defined by Atiyah [1] in this analytic context to be $\widetilde{F}(0)$. The equality of the combinatorial and analytic versions was proven by Dodziuk [10].

It was shown in [16] that the analytic and combinatorial spectral density functions are dilation equivalent. That is, there is some $K > 0$ so that

$$F_j(K^{-1}\lambda) \leq \widetilde{F}_j(\lambda) \leq F_j(K\lambda)$$

for all $\lambda \geq 0$. We will make use of this result in Section 4.2.

Say that Y is of *analytic determinant class* if the integral

$$\int_{0+}^{1} \log \lambda d\widetilde{F}_j(\lambda)$$

exists for all j. We can also ask if Y is of determinant class in the combinatorial sense defined earlier, by choosing a triangulation of Y/π. It was observed in [5] (using the dilation equivalence) that this question is independent of triangulation, and equivalent to analytic determinant class.

The analytic definition of L^2-torsion is tricky, as the analytic Laplacian is no longer a bounded operator, and was first carried out independently by Lott [20] and Mathai [25]. In [6], results of [5] are interpreted as the equality of the combinatorial and analytic L^2-torsions. This makes the question of homotopy invariance of combinatorial L^2-torsion especially interesting, because of the implications for L^2-analytic torsion.

Specifically, suppose Y is a symmetric space, and consider any operator T on $L^2\Lambda^j(Y)$ which is invariant under the full isometry group of Y. The local trace of its kernel, $\mathrm{tr}\, T(x,x)$, is a constant function, so that $\mathrm{Tr}_\pi T = \mathrm{Vol}(\mathcal{F}) \cdot C(T)$ for some $C(T)$ which does not depend on π. In particular, $\widetilde{\Delta}$ and its spectral projections are all invariant under isometries, so that the L^2-analytic torsion of Y is given by $C \cdot \mathrm{Vol}(\mathcal{F})$, where $C = C(Y)$ depends only on Y (but may be zero).

If one can show $C \neq 0$, the homotopy invariance of L^2-torsion gives a rigidity result which says that π can only occur as a lattice of Y with some fixed covolume. However, in the situations where it is possible to establish nontriviality of L^2-torsion, one typically knows stronger rigidity results. One example is the case of odd dimensional hyperbolic space H^{2n+1}, where one has Mostow rigidity. Lott [20] provides a method for computing $C(H^{2n+1})$ for any n, and Hess and Schick [17] have used that to show that $C(H^{2n+1}) > 0$ for all n.

CHAPTER 2

PROPERTIES OF GROUPS

2.1. Residual Properties

Definition 2.1. Let $\mathcal{C}$ be a nonempty class of groups (though possibly containing only one group). A group π is *residually* $\mathcal{C}$ if for any element $g \in \pi$, $g \neq e$, there exists a quotient group $\pi'(g)$ belonging to $\mathcal{C}$ such that $g \mapsto g' \in \pi'(g)$ with $g' \neq e$.

If a countable group π is residually $\mathcal{C}$, then there is a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that π/Γ_n belongs to $\mathcal{C}$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. For other basic theorems concerning residual properties of groups, we refer to [24].

When $\mathcal{C} = \{\text{finite groups}\}$ we say that π is a *residually finite* group.

2.2. Amenability

Let π be a finitely generated discrete group, with word metric d. We use the following characterization of amenability, due to Følner.

Definition 2.2. π is *amenable* if there is a sequence of finite subsets $\{\Lambda_k\}_{k=1}^{\infty}$ such that for any fixed $\delta > 0$

$$\lim_{k \to \infty} \frac{\#\{\partial_\delta \Lambda_k\}}{\#\{\Lambda_k\}} = 0$$

where $\partial_\delta \Lambda_k = \{\gamma \in \pi : d(\gamma, \Lambda_k) < \delta \text{ and } d(\gamma, \pi - \Lambda_k) < \delta\}$ is a δ-neighborhood of the boundary of Λ_k.

Examples of amenable groups include finite groups, abelian groups, nilpotent groups and solvable groups, and groups of subexponential growth. Amenability for discrete groups is preserved by the following five processes [27, Prop. 0.16]:

1. Taking subgroups;

2. Forming quotient groups;
3. Forming group extensions by amenable groups;
4. Forming upward directed unions of amenable groups;
5. Forming a direct limit of amenable groups.

Free groups with two or more generators, and fundamental groups of closed negatively curved manifolds are *not* amenable [14].

2.3. Residual Amenability

Definition 2.3. If π is residually $\mathcal{C}$, where $\mathcal{C} = \{\text{amenable groups}\}$, we say that π is *residually amenable.*

Recall that the derived subgroups $\pi^{(i)}$ of a group π are defined by $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. Say that π is *solvable* if $\pi^{(i)} = \{e\}$ for some i. The *rank* of π is then defined to be the smallest i for which $\pi^{(i)} = \{e\}$.

Contained in the class of residually amenable groups is the class of residually solvable groups. Residually solvable groups are naturally characterized in Prop 2.1. Free products of residually solvable groups are themselves residually solvable. This follows from the fact that solvability is a root property as discussed in [24]. An interesting question is whether amenability is a root property.

Proposition 2.1. *π is residually solvable if and only if $\bigcap_{i=1}^{\infty} \pi^{(i)} = \{e\}$.*

Proof. Since $\pi/\pi^{(i)}$ is solvable, if $\bigcap_{i=1}^{\infty} \pi^{(i)} = \{e\}$ then π is residually solvable. Now suppose π is residually solvable, and let $g \in \bigcap_{i=1}^{\infty} \pi^{(i)}$. For any map $f : \pi \to S$ with S solvable of rank k, we have $f(g) \in f(\pi^{(i)}) \subseteq S^{(i)} = \{e\}$ for $i > k$. Since the image of g is trivial in any solvable quotient of π we must have $g = e$. □

Proposition 2.2. *If Γ is residually solvable, S is solvable, and π is an extension*

$$1 \to \Gamma \xrightarrow{\iota} \pi \xrightarrow{\kappa} S \to 1$$

then π is residually solvable.

Proof. Suppose S has rank k. Then $\kappa(\pi^{(k)}) \subseteq S^{(k)} = \{e\}$, so $\pi^{(k)} \subseteq \Gamma$ and therefore $\pi^{(k+i)} \subseteq \Gamma^{(i)}$ for all $i \geq 0$. Then $\bigcap_{i=1}^{\infty} \pi^{(i)} \subseteq \bigcap_{i=1}^{\infty} \Gamma^{(i)} = \{e\}$. □

Example 2.1. For nonzero integers p and q, define the *Baumslag-Solitar* group $BS(p, q)$ by

$$BS(p, q) = \left\langle a, b \mid a^{-1}b^p a = b^q \right\rangle.$$

A group π is *Hopfian* if $\pi/\Gamma \cong \pi$ implies $\Gamma = \{e\}$. The family of groups $BS(p, q)$ were first defined in [2], where it was shown that $BS(p, q)$ is Hopfian if and only if p and q are *meshed*, which means $p|q$, $q|p$, or p and q have exactly the same set of prime divisors. As any finitely generated residually finite group is Hopfian, the groups $BS(p, q)$ are not residually finite when p and q are not meshed.

Kropholler shows in [19] that the second derived subgroup $\pi^{(2)}$ is free when $\pi = BS(p, q)$, for any p, q. When p and q are not both ± 1, $\pi^{(2)}$ is free on two or more generators and therefore π is not amenable. However, π is residually solvable since it is an extension

$$1 \to \pi^{(2)} \to \pi \to \pi/\pi^{(2)} \to 1$$

with $\pi^{(2)}$ residually solvable and $\pi/\pi^{(2)}$ solvable (of rank 2).

More generally, let π be any non-cyclic group which is the fundamental group of a graph of infinite cyclic groups. Then from [19], $\pi^{(2)}$ is free and the above argument shows π is residually solvable.

Example 2.2. We show that the amalgamated free product of two abelian groups is residually solvable. Slightly more generally, suppose we have two solvable groups A and B, and two monomorphisms from an abelian group H into the centers of A and B given by $\alpha : H \to Z(A)$ and $\beta : H \to Z(B)$. Then the free product with amalgamation $A *_H B$ is residually solvable.

To see this, let $N = \{(\alpha(h), \beta(h)^{-1}) | h \in H\} \subset A \times B$. Since H is abelian, N is a subgroup of $A \times B$. N is normal because H includes into the centers of both A

and B. Note that if H is only known to be normal in A and B, N is unlikely to be normal in $A \times B$.

Let K be the kernel of the natural map $A *_H B \to (A \times B)/N$. In $A *_H B$, K has trivial intersection with all conjugates of A and B, hence K is free by a well known theorem of group actions on trees (see for example [3, pg 54]). Then $A *_H B$ is an extension of the solvable group $(A \times B)/N$ by the residually solvable group K, and so $A *_H B$ is residually solvable.

Example 2.3. It is shown in [28] that any HNN-extension of a finitely generated abelian group is residually solvable.

Example 2.4. R.J. Thompson, G. Higman, K. Brown, and E.A. Scott have produced various classes of finitely presented infinite simple groups (see [29]). The example of Scott contains a free subgroup on two generators and is therefore not amenable and not residually amenable.

CHAPTER 3

MAIN TECHNICAL THEOREMS

3.1. Approximating The Spectrum Of Δ

As in the introduction, suppose Y is a simplicial complex, π acts freely and simplicially on Y, and $X = Y/\pi$ is a finite simplicial complex. Suppose we have a nested sequence of normal subgroups $\pi = \Gamma_1 \supset \Gamma_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$. Define $Y_n = Y/\Gamma_n$.

Suppose X has a_j cells in dimension j, and choose a lift to Y of each j-cell of X. These choices give a basis over $l^2(\pi)$ of the space $C^j_{(2)}(Y)$ of j dimensional l^2-cochains on Y. The lifts also descend to give bases of $C^j_{(2)}(Y_n)$ over $l^2(\pi/\Gamma_n)$.

Denote by Δ and Δ_n the Laplacian on $C^j_{(2)}(Y)$ and $C^j_{(2)}(Y_n)$ respectively. All arguments to follow will apply to a specific value of j, but this dependence will not be indicated.

Let $\{E(\lambda) : \lambda \in [0,\infty)\}$ and $\{E_n(\lambda) : \lambda \in [0,\infty)\}$ denote the right continuous family of spectral projections of Δ and Δ_n. Since Δ is π-equivariant, so are $E(\lambda) = \chi_{[0,\lambda]}(\Delta)$ for $\lambda \in [0,\infty)$. Similarly, $E_n(\lambda)$ are π/Γ_n-equivariant. Let $F, F_n : [0,\infty) \to [0,\infty)$ denote the spectral density functions

$$F(\lambda) = \mathrm{Tr}_\pi \, E(\lambda)$$

$$F_n(\lambda) = \mathrm{Tr}_{\pi/\Gamma_n} \, E_n(\lambda).$$

We now set

$$\begin{aligned} \overline{F}(\lambda) &= \limsup_{n\to\infty} F_n(\lambda); & \underline{F}(\lambda) &= \liminf_{n\to\infty} F_n(\lambda) \\ \overline{F}^+(\lambda) &= \lim_{\delta\to 0^+} \overline{F}(\lambda+\delta); & \underline{F}^+(\lambda) &= \lim_{\delta\to 0^+} \underline{F}(\lambda+\delta). \end{aligned}$$

With the above notation, we state the main technical results of this paper.

Theorem 3.1. *For all* $\lambda \in [0,\infty)$,

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda)$$

Theorem 3.2. *Suppose there is some right continuous function* $s : [0,\varepsilon) \to [0,\infty)$, *with* $s(0) = 0$ *and so that for all* n *and for all* $\lambda \in [0,\varepsilon)$ *we have*

$$F_n(\lambda) - F_n(0) \leq s(\lambda)$$

then

1. $\overline{F}(\lambda)$ *and* $\underline{F}(\lambda)$ *are right continuous at zero and one has the equalities*

$$\overline{F}(0) = \overline{F}^+(0) = F(0) = \underline{F}(0) = \underline{F}^+(0).$$

2. *For all* $\lambda \in [0,\varepsilon)$,

$$F(\lambda) - F(0) \leq s(\lambda).$$

These theorems and their proofs are similar to Lück [21, Theorem 2.3], but here they are stated so as to require no conditions on the quotient groups π/Γ_n. The residual finiteness condition in Lück's Theorem or the residual amenability condition of this paper are required to provide s, the uniform decay at zero of the spectral density functions for the covers Y_n.

To show the two technical theorems, we first prove a number of preliminary lemmas.

Lemma 3.1. *There exists a number* $K > 1$ *such that the operator norms of* Δ *and* Δ_n *are smaller than* K *for all* n.

Proof. Choosing lifts of cells of X, we have identified the space of l^2-cochains on Y with $\bigoplus_{i=1}^a l^2(\pi)$. The combinatorial Laplacian Δ is then described by an $a \times a$ matrix B with entries in $\mathbb{Z}[\pi]$, acting by right multiplication. The Laplacian Δ_n is described by the same matrix B, now acting by right multiplication on $\bigoplus_{i=1}^a l^2(\pi/\Gamma_n)$

For $u = \sum_{g\in\pi} \lambda_g \cdot g \in \mathbb{C}[\pi]$ define $|u|_1 = \sum_{g\in\pi} |\lambda_g|$. Notice that

$$|ux| = \left|\sum_{g\in\pi} \lambda_g \cdot g \cdot x\right| \leq \sum_{g\in\pi} |\lambda_g| \cdot |g \cdot x| = \sum_{g\in\pi} |\lambda_g| \cdot |x| \leq |u|_1 \cdot |x| \,.$$

Now choose $K > 1$ so that

$$K \geq a \cdot \sum_{j=1}^{b} \max \left\{ |B_{ij}|_1 \,; i = 1, \ldots, a \right\}.$$

We now show that $\|\Delta\| \leq K$. The proof for Δ_n is completely analogous. Let $x = (x_1, x_2, \ldots, x_a) \in \bigoplus_{i=1}^{a} l^2(\pi)$. Then

$$\begin{aligned}
|xB|^2 &= \sum_{j=1}^{a} \left| \sum_{i=1}^{a} x_i \cdot B_{ij} \right|^2 \\
&\leq \sum_{j=1}^{a} \left(\sum_{i=1}^{a} |x_i \cdot B_{ij}| \right)^2 \\
&\leq \sum_{j=1}^{a} \left(\sum_{i=1}^{a} |x_i| \cdot |B_{ij}|_1 \right)^2 \\
&\leq \sum_{j=1}^{a} \left(\sum_{i=1}^{a} |x_i| \cdot \max \left\{ |B_{ij}|_1 \,; i = 1, \ldots, a \right\} \right)^2 \\
&\leq \sum_{j=1}^{a} \left(\max \left\{ |B_{ij}|_1 \,; i = 1, \ldots, a \right\} \right)^2 \cdot \left(\sum_{i=1}^{a} |x_i| \right)^2 \\
&\leq \sum_{j=1}^{a} \left(\max \left\{ |B_{ij}|_1 \,; i = 1, \ldots, a \right\} \right)^2 \cdot a^2 \cdot \sum_{i=1}^{a} |x_i|^2 \\
&\leq \left(a \cdot \sum_{j=1}^{a} \max \left\{ |B_{ij}|_1 \,; i = 1, \ldots, a \right\} \right)^2 \cdot |x|^2
\end{aligned}$$

which finishes the proof of Lemma 3.1. □

Lemma 3.2. *Let $p(\mu)$ be a polynomial. There is a number n_0, depending only on the system of groups $\pi = \Gamma_1 \supset \Gamma_2 \supset \ldots$ such that for all $n \geq n_0$*

$$\operatorname{Tr}_\pi p(\Delta) = \operatorname{Tr}_{\pi/\Gamma_n} p(\Delta_n)$$

Proof. We identify Δ with an $a \times a$ matrix B with entries in $\mathbb{Z}[\pi]$, as in the previous Lemma.

Fix elements $g_0, g_1, \dots, g_r \in \pi$ and $\lambda_0, \lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that $g_0 = e$, $g_i \neq e$, and $\lambda_i \neq 0$ for $1 \leq i \leq r$ so that

$$\sum_{j=1}^{a} (p(B))_{j,j} = \sum_{i=0}^{r} \lambda_i g_i.$$

Then

$$\mathrm{Tr}_\pi \, p(\Delta) = \lambda_0.$$

The Laplacian Δ_n on Y_n is also described by the matrix B, now acting on $\bigoplus_{i=1}^a l^2(\pi/\Gamma_n)$ by right multiplication. Then

$$\mathrm{Tr}_{\pi/\Gamma_n} \, p(\Delta_n) = \sum_{i=1}^{r} \lambda_i \, \mathrm{Tr}_{\pi/\Gamma_n} \, R(g_i)$$

where $R(g_i) : l^2(\pi/\Gamma_n) \to l^2(\pi/\Gamma_n)$ is right multiplication with g_i.

Since the intersection of the Γ_i's is trivial, there is a number n_0 such that for $n \geq n_0$ none of the elements g_i for $1 \leq i \leq r$ lies in Γ_n. Since Γ_n is normal, we conclude for $n \geq n_0$ and $i \neq 0$

$$\mathrm{Tr}_{\pi/\Gamma_n} \, R(g_i) = 0.$$

Then for $n \geq n_0$

$$\mathrm{Tr}_\pi \, p(\Delta) = \lambda(0) = \mathrm{Tr}_{\pi/\Gamma_n} \, p(\Delta_n).$$

□

Lemma 3.3. *Let K be as in 3.1. Let $\{p_k(\mu)\}_{k=1}^\infty$ be a sequence of polynomials, uniformly bounded on $[0, K]$, such that for the characteristic function $\chi_{[0,\lambda]}(\mu)$ of the interval $[0, \lambda]$,*

$$\lim_{k\to\infty} p_k(\mu) = \chi_{[0,\lambda]}(\mu)$$

holds for each $\mu \in [0, K]$. Then

$$\lim_{k\to\infty} \mathrm{Tr}_\pi \, p_k(\Delta) = F(\lambda).$$

Proof. This proof of this lemma is the same as [21, Lemma 2.7]. As before, let $\{E(\lambda) : \lambda \in [0, \infty)\}$ denote the spectral family for Δ . Then

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Tr}_\pi\, p_n(\Delta) &= \lim_{n\to\infty} \mathrm{Tr}_\pi \left(\int_0^K p_n(\lambda) dE(\lambda) \right) \\ &= \lim_{n\to\infty} \int_0^K p_n(\lambda) dF(\lambda) \\ &= \int_0^K \chi_{[0,\lambda]} dF(\lambda) \\ &= F(\lambda).\end{aligned}$$

The second equality follows from the fact that the Von Neumann trace is linear, monotone, and ultraweakly continuous. The third inequality is Lebesgue's Theorem of Majorized Convergence. □

We now prove Theorem 3.1. Fix $\lambda \geq 0$. Define for $k \geq 1$ a continuous function $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(\mu) = \begin{cases} 1 + \frac{1}{k} & \mu \leq \lambda \\ 1 + \frac{1}{k} - k(\mu - \lambda) & \lambda \leq \mu \leq \lambda + \frac{1}{k} \\ \frac{1}{k} & \lambda + \frac{1}{k} \leq \mu \end{cases}$$

Clearly $\chi_{[0,\lambda]}(\mu) < f_{k+1}(\mu) < f_k(\mu)$ and $f_k(\mu)$ converges to $\chi_{[0,\lambda]}(\mu)$ for all $\mu \in [0, \infty)$. For each k choose a polynomial p_k such that $\chi_{[0,\lambda]}(\mu) < p_k(\mu) < f_k(\mu)$ holds for all $\mu \in [0, K]$, where K is as in Lemma 3.1. The polynomials p_k satisfy the conditions of Lemma 3.3.

Because $\chi_{[0,\lambda]}(\mu) \leq p_k(\mu)$ for all $\mu \in [0, \|\Delta_n\|]$, we have

$$\begin{aligned}F_n(\lambda) &= \mathrm{Tr}_{\pi/\Gamma_n}\big(\chi_{[0,\lambda]}(\Delta_n)\big) \\ &\leq \mathrm{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big).\end{aligned} \tag{3.1}$$

On the other hand, we have $p_k(\mu) \leq 1 + \frac{1}{k}$ for $\mu \in [0, \lambda + \frac{1}{k}]$ and $p_k(\mu) \leq \frac{1}{k}$ for

$\mu \in [\lambda + \frac{1}{k}, K]$. So

$$\begin{aligned} \mathrm{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big) &\le \mathrm{Tr}_{\pi/\Gamma_n}\big((1+\tfrac{1}{k})\chi_{\left[0,\lambda+\frac{1}{k}\right]}(\Delta_n)\big) \\ &\quad + \mathrm{Tr}_{\pi/\Gamma_n}\big((\tfrac{1}{k})\chi_{\left[\lambda+\frac{1}{k},\,K\right]}(\Delta_n)\big) \\ &= (1+\tfrac{1}{k})F_n(\lambda+\tfrac{1}{k}) \\ &\quad + \tfrac{1}{k}(F_n(K) - F_n(\lambda+\tfrac{1}{k})) \\ &= F_n(\lambda+\tfrac{1}{k}) + \tfrac{1}{k}F_n(K) \end{aligned} \tag{3.2}$$

Now notice $F_n(K) = \mathrm{Tr}_{\pi/\Gamma_n}\big(\chi_{[0,K]}(\Delta_n)\big)$. But $\chi_{[0,K]}(\Delta_n)$ is the identity on the space $C^j_{(2)}(Y_n)$, which is identified with $\bigoplus_{i=1}^{a_j} l^2(\pi/\Gamma_n)$. Thus

$$F_n(K) = a_j. \tag{3.3}$$

By Lemma 3.2, there is a number $n_0(k)$ for each polynomial p_k such that for $n \ge n_0(k)$

$$\mathrm{Tr}_\pi\, p_k(\Delta) = \mathrm{Tr}_{\pi/\Gamma_n}\big(p_k(\Delta_n)\big).$$

Then for $n \ge n_0(k)$, the equations (3.1), (3.2), and (3.3) give

$$F_n(\lambda) \le \mathrm{Tr}_\pi\, p_k(\Delta) \le F_n(\lambda + \tfrac{1}{k}) + \tfrac{1}{k}a_j$$

Taking limits as $n \to \infty$,

$$\overline{F}(\lambda) \le \mathrm{Tr}_\pi\, p_k(\Delta) \le \underline{F}(\lambda + \tfrac{1}{k}) + \tfrac{1}{k}a_j$$

Taking limits as $k \to \infty$, and using Lemma 3.3,

$$\overline{F}(\lambda) \le F(\lambda) \le \underline{F}^+(\lambda).$$

We have for all $\epsilon > 0$

$$F(\lambda) \le \underline{F}^+(\lambda) \le \underline{F}(\lambda+\epsilon) \le \overline{F}(\lambda+\epsilon) \le F(\lambda+\epsilon)$$

and since $\lim_{\epsilon\to 0+} F(\lambda+\epsilon) = F(\lambda)$ we get

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda).$$

This finishes the proof of Theorem 3.1. □

Next we show Theorem 3.2. We suppose there is some right continuous function $s : [0, \varepsilon) \to [0, \infty)$, with $s(0) = 0$ and so that for all n and for all $\lambda \in [0, \varepsilon)$ we have

$$F_n(\lambda) - F_n(0) \leq s(\lambda).$$

Taking the limit inferior and limit superior for $n \to \infty$ gives:

$$\underline{F}(\lambda) \leq \underline{F}(0) + s(\lambda) \text{ and } \overline{F}(\lambda) \leq \overline{F}(0) + s(\lambda).$$

Taking the limit for $\lambda \to 0$ gives

$$\underline{F}^+(0) \leq \underline{F}(0) \text{ and } \overline{F}^+(0) \leq \overline{F}(0).$$

And finally, since $\underline{F}$ and $\overline{F}$ are increasing,

$$\underline{F}^+(0) = \underline{F}(0) \text{ and } \overline{F}^+(0) = \overline{F}(0).$$

We already know $\overline{F}^+(0) = F(0) = \underline{F}^+(0)$ from Theorem 3.1, and this proves Theorem 3.2.1. Since s is right continuous, we conclude:

$$\overline{F}^+(\lambda) \leq F(0) + s(\lambda)$$

and Theorem 3.2.2 follows from Theorem 3.1. This finishes the proof of Theorem 3.2. □

3.2. Convergence Lemmas

This section contains some technical convergence lemmas needed in Chapter 4.

Definition 3.1. A *density function* is a non-decreasing function $F : [0, \infty) \to [0, \infty)$. Given a density function F, define the *right continuous density function* F^+ by

$$F^+(\lambda) = \lim_{\delta \to 0^+} F(\lambda + \delta).$$

Define the *Betti number* of F by

$$b(F) = \lim_{\delta \to 0^+} F(\delta) = F^+(0).$$

If there is some K so that $F(\lambda) = F(K)$ for $\lambda \geq K$, put

$$\log \det(F) = \int_{0+}^{K} \log(\lambda) dF(\lambda) \tag{3.4}$$

if the Lebesgue-Stieltjes integral on the right exists (compare 1.1).

Attempting to integrate (3.4) by parts leads to expressions of the form

$$\int_{0^+}^{K} \frac{F(\lambda) - b(F)}{\lambda} d\lambda$$

so we state lemmas governing convergence of such expressions. First, making F right continuous has no effect. Specifically, Lück [21, Lemma 3.2] shows that for any $K > 0$,

$$\int_{0^+}^{K} \frac{F(\lambda) - b(F)}{\lambda} d\lambda = \int_{0^+}^{K} \frac{F^+(\lambda) - b(F^+)}{\lambda} d\lambda. \tag{3.5}$$

We now consider the specific situation where F and F_n are the spectral density functions for Δ and Δ_n. As these are right continuous, $b(F) = F(0)$, $b(F_n) = F_n(0)$. The following pair of Lemmas are proved in [21, Lemma 3.3]. Here, their proofs will only be sketched.

Lemma 3.4. *Suppose* $\lim_{n\to\infty} F_n(0) = F(0)$ *(which follows from Theorem 3.2 when the decay hypothesis is satisfied). Then*

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq \liminf_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

Proof. First, using Theorem 3.1,

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda = \int_{0^+}^{K} \liminf_{n\to\infty} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

because $F_n(0) \to F(0)$ and the function F is exactly the liminf of F_n made right continuous. From (3.5), the right continuity has no effect on the integral.

Second, the fact that

$$\int_{0^+}^{K} \liminf_{n\to\infty} \frac{F_n(\lambda)-F_n(0)}{\lambda} d\lambda \leq \liminf_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda)-F_n(0)}{\lambda} d\lambda$$

is quite general, and follows from the monotone convergence theorem. A sufficient condition for the interchange of liminf and integral is that the functions $\frac{F_n(\lambda)-F_n(0)}{\lambda}$ are uniformly bounded from below (by zero). □

Lemma 3.5. *Suppose the F_n have a uniform spectral gap, that is there is some $\epsilon > 0$ so that $F_n(\lambda) = F_n(0)$ for all n and for all $\lambda \leq \epsilon$. Then*

$$\int_{0^+}^{K} \frac{F(\lambda)-F(0)}{\lambda} d\lambda \geq \limsup_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda)-F_n(0)}{\lambda} d\lambda$$

and therefore using the previous Lemma,

$$\int_{0^+}^{K} \frac{F(\lambda)-F(0)}{\lambda} d\lambda = \lim_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda)-F_n(0)}{\lambda} d\lambda.$$

Proof. The proof here is nearly identical to the previous Lemma 3.4. However, the gap hypothesis is required for the interchange of an integral and limsup. The uniform spectral gap easily gives the sufficient condition that the functions $\frac{F_n(\lambda)-F_n(0)}{\lambda}$ are uniformly bounded from above.

Lück gives a sharper condition than simply demanding a spectral gap, but it will not be needed. □

CHAPTER 4

PROOFS OF THE MAIN THEOREMS

In this chapter, we prove the approximation theorem, determinant class theorem, and homotopy invariance of L^2-torsion for residually amenable groups.

4.1. The Approximation Theorem

Proof of Theorem 0.1 (Approximation Theorem).

Observe that the jth L^2-Betti numbers of Y and Y_n are given by

$$b^j_{(2)}(Y : \pi) = F(0); \quad b^j_{(2)}(Y_n : \pi/\Gamma_n) = F_n(0).$$

Therefore Theorem 0.1 will follow directly from Theorem 3.2.1, if we can establish a uniform decay of F_n near zero.

Since π/Γ_n is amenable, [11, Theorem 2.1.3] applies, and we have a constant $K > 1$ so that

$$F_n(\lambda) - F_n(0) \leq a_j \frac{\log K}{-\log \lambda} = s(\lambda) \tag{4.1}$$

for all $0 < \lambda < 1$. The constant K can be any number larger than $\max(\|\Delta_n\|, 1)$, and therefore can be chosen independently of n, by Lemma 3.1. $\square$

Remark. In the case of Lück's Theorem, the groups π/Γ_n are finite, so the Laplacian on Y_n is a finite, self-adjoint matrix. Lück then proves, for any self-adjoint matrix, an elegant estimate on the number of eigenvalues which are less than a fixed λ. The estimate is weakened if the product of the nonzero eigenvalues is small, but this product must be at least one as the Laplacian has integer entries.

Dodziuk and Mathai make use of the same fundamental estimate when proving the approximation theorem for amenable groups.

4.2. Results For Manifolds

Suppose M is a compact Riemannian manifold, $\widetilde{M}$ is a regular covering space for M with residually amenable transformation group, and let $\widetilde{\Delta}$ denote the Laplacian on L^2 j-forms on $\widetilde{M}$. Following arguments in [11], one can investigate the analytic spectral density function $\widetilde{F}(\lambda)$ of $\widetilde{\Delta}$. To relate this analytic Laplacian to the combinatorial situation in the previous sections, choose X to be a triangulation of M and lift to get a triangulation Y of $\widetilde{M}$, as in Section 1.4.

The spectrum of $\widetilde{\Delta}$ is said to have a *gap at zero* if the spectral projection $\widetilde{E}(\lambda) = \widetilde{E}(0)$ for some $\lambda > 0$. Because the von Neumann dimension is faithful, $\widetilde{\Delta}$ has a spectral gap at zero if and only if $\widetilde{F}(\lambda) = \widetilde{F}(0)$ for some $\lambda > 0$.

Theorem 4.1. *(Gap Criterion) The spectrum of $\widetilde{\Delta}$ has a gap at zero if and only if there is a $\lambda > 0$ such that*

$$\lim_{n\to\infty} F_n(\lambda) - F_n(0) = 0.$$

Proof. Since $\widetilde{F}$ and F are dilation equivalent, $\widetilde{\Delta}$ will have a spectral gap if and only if Δ does. By Theorem 3.1, $F(\lambda) = F(0)$ if and only if

$$\lim_{n\to\infty} F_n(\lambda) - F_n(0) = 0.$$

□

Theorem 4.2. *(Spectral Density Estimate) There are constants $C > 0$ and $\varepsilon > 0$ independent of λ, such that for all $\lambda \in (0, \varepsilon)$*

$$\widetilde{F}(\lambda) - \widetilde{F}(0) \le \frac{C}{-\log \lambda}.$$

Proof. This follows directly from dilation equivalence and the estimate 4.1. □

The spectral density estimate is interesting as it provides evidence supporting the well known conjecture that the Novikov-Shubin invariants of a closed manifold are positive.

4.3. The Determinant Class Theorem

Recall that a covering space Y of a finite simplicial complex X is of determinant class if for each j,

$$-\infty < \int_{0^+}^{1} \log \lambda dF_j(\lambda),$$

where $F_j(\lambda)$ denotes the von Neumann spectral density function of the combinatorial Laplacian Δ_j on L^2 j-cochains.

We will prove that every residually amenable covering of a finite simplicial complex is of determinant class. The appendix of [4] contains a proof that every residually finite covering of a compact manifold is of determinant class. Their proof is based on Lück's approximation of von Neumann spectral density functions. Since an analogous approximation holds in the setting of this paper, we can apply the argument of [4] to prove Theorem 0.2. The fact that our coverings are infinite necessitates some modifications.

Proof of Theorem 0.2 (Determinant Class Theorem).

As with the rest of this paper, this proof will proceed for a fixed j which will be suppressed in the notation.

Denote by $\operatorname{Det}_{\pi/\Gamma_n} \Delta_n'$ the Fuglede-Kadison determinant of Δ_n restricted to the orthogonal complement of its kernel. It is given by the following Lebesgue-Stieltjes integral,

$$\log \operatorname{Det}_{\pi/\Gamma_n} \Delta_n' = \int_{0^+}^{\infty} \log \lambda dF_n(\lambda) = \int_{0^+}^{K} \log \lambda dF_n(\lambda)$$

with K as in Lemma 3.1. That is, $\|\Delta_n\| \leq K$.

Integration by parts yields

$$\begin{aligned} \log \operatorname{Det}_{\pi/\Gamma_n} \Delta_n' &= (\log K)\big(F_n(K) - F_n(0)\big) \\ &\quad + \lim_{\epsilon \to 0^+} \left\{ -(\log \epsilon)\big(F_n(\epsilon) - F_n(0)\big) - \int_{\epsilon}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \right\}. \end{aligned} \tag{4.2}$$

Now Y_n is an amenable cover of X, so we are in the situation of Dodziuk-Mathai [11]. We require two results contained in their proof of the Determinant Class Theorem for amenable coverings [11, Thm 0.2]. The first is that

$$\log \operatorname{Det}_{\pi/\Gamma_n} \Delta'_n \geq 0,$$

which is stronger than simply determinant class - the key point being that the bound is uniform in n. The second is the existence of the integral

$$\int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda. \tag{4.3}$$

It is worth remarking that their proof and the corresponding proof in [4] require similar statements for the Laplacians on finite approximations. In the finite case, the Laplacian is an integer matrix, so the product of its nonzero eigenvalues is a positive integer and therefore at least 1. The existence of the integral (4.3) is clear in the finite case, as the spectrum is discrete.

From (4.2) and the existence of (4.3),

$$L = - \lim_{\epsilon \to 0^+} (\log \epsilon)\big(F_n(\epsilon) - F_n(0)\big)$$

must exist. In fact $L = 0$, since if not we can fix $0 < \epsilon < L$ and then

$$-(\log \lambda)\big(F_n(\lambda) - F_n(0)\big) > L - \epsilon > 0$$

and

$$F_n(\lambda) - F_n(0) > \frac{L - \epsilon}{-\log(\lambda)}$$

for small λ, which contradicts the existence of (4.3). In summary,

$$\lim_{\epsilon \to 0^+} (\log \epsilon)\big(F_n(\epsilon) - F_n(0)\big) = 0.$$

Now (4.2) becomes

$$\log \operatorname{Det}_{\pi/\Gamma_n} \Delta'_n = (\log K)\big(F_n(K) - F_n(0)\big) - \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda$$

which gives

$$\int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \leq (\log K)\big(F_n(K) - F_n(0)\big).$$

Now from Lemma 3.4,

$$\begin{aligned} \int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda &\leq \liminf_{n\to\infty} \int_{0^+}^{K} \frac{F_n(\lambda) - F_n(0)}{\lambda} d\lambda \\ &\leq \liminf_{n\to\infty} (\log K)\big(F_n(K) - F_n(0)\big). \end{aligned} \tag{4.4}$$

Since we have uniform decay (4.1) of the F_n near 0, Theorem 3.2.1 applies and $\lim_{n\to\infty} F_n(0) = F(0)$. Since $K \geq \|\Delta_n\|$ for all n and $K \geq \|\Delta\|$,

$$F_n(K) = F(K) = a_j$$

for all n, so (4.4) becomes

$$\int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \leq (\log K)\big(F(K) - F(0)\big). \tag{4.5}$$

This shows in particular that the left hand integral exists, and arguing as with Δ_n we have

$$\log \mathrm{Det}_\pi \Delta' = (\log K)\big(F(K) - F(0)\big) - \int_{0^+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda$$

which is non-negative by (4.5). Since this is true for all j, Y is of determinant class.

□

4.4. Homotopy Invariance of L^2-Torsion

In this section, we focus on closed, compact, odd dimensional manifolds, so that the Euler characteristic vanishes and therefore L^2-torsion is a topological invariant. Note that for even dimensional L^2-acyclic manifolds of determinant class, the L^2-torsion vanishes [25].

The Fuglede-Kadison determinant Det_π induces a homomorphism from the Whitehead group of π,

$$\Phi_\pi : \mathrm{Wh}(\pi) \to \mathbb{R}^+$$

which was defined in [23] and [21]. Given a homotopy equivalence $f : M \to N$ of compact manifolds, we choose cell decompositions for M and N, choose f to be a cellular homotopy equivalence, and let M_f be the cellular mapping cone. Putting $\pi = \pi_1(M)$, the cochain complex $C^*(M_f)$ is an acyclic complex over the group ring $\mathbb{Z}[\pi]$, and defines the Whitehead torsion $T(f) \in \mathrm{Wh}(\pi)$. Then $\Phi_\pi(T(f)) \in \mathbb{R}^+$.

Now suppose π is residually amenable, so M and N are of determinant class. Let $\phi_{\widetilde{M}}$, $\phi_{\widetilde{N}}$ denote the L^2-torsion of M and N. The homotopy equivalence f canonically identifies the determinant lines of L^2-cohomology of $\widetilde{M}$ and $\widetilde{N}$, so $\phi_{\widetilde{M}} \otimes \phi_{\widetilde{N}}^{-1} \in \mathbb{R}^+$.

Then from [26, Prop. 2.1], one has

$$\phi_{\widetilde{M}} \otimes \phi_{\widetilde{N}}^{-1} = \Phi_\pi(T(f)) \in \mathbb{R}^+.$$

Therefore, Theorem 0.3 is reduced to the following

Theorem 4.3. *Suppose that π is a finitely presented residually amenable group. Then the homomorphism*

$$\Phi_\pi : \mathrm{Wh}(\pi) \to \mathbb{R}^+$$

is trivial.

Proof. We can represent an arbitrary element of $\mathrm{Wh}(\pi)$ as the Whitehead torsion of a homotopy equivalence $f : L \to K$ of finite CW-complexes, which without loss of generality is an inclusion. Let $\widetilde{L}$ and $\widetilde{K}$ denote the corresponding regular π covering complexes. The relative cochain complex $C^*(\widetilde{K}, \widetilde{L})$ is acyclic, and so is its L^2-completion $C^*_{(2)}(\widetilde{K}, \widetilde{L})$. In particular, the combinatorial Laplacian $\Delta_j^{\widetilde{K},\widetilde{L}}$ is invertible and we see that

$$\Phi_\pi(T(f)) = \prod_{j=0}^{n} \mathrm{Det}_\pi(\Delta_j^{\widetilde{K},\widetilde{L}})^{\frac{(-1)^j j}{2}} > 0.$$

We claim that $\mathrm{Det}_\pi(\Delta_j^{\widetilde{K},\widetilde{L}}) = 1$ for each j. The j will be suppressed in the notation.

Form the covering spaces $\widetilde{L}_n$ and $\widetilde{K}_n$ with covering group π/Γ_n, and let $\Delta^{\widetilde{K}_n,\widetilde{L}_n}$ denote the Laplacian on $C^*_{(2)}(\widetilde{K}_n, \widetilde{L}_n)$. As $\Delta^{\widetilde{K},\widetilde{L}}$ is invertible we can choose a common bound K on $\left\|\Delta^{\widetilde{K},\widetilde{L}}\right\|$ and $\left\|(\Delta^{\widetilde{K},\widetilde{L}})^{-1}\right\|$, as in Lemma 3.1. K will also bound $\left\|\Delta^{\widetilde{K}_n,\widetilde{L}_n}\right\|$ and $\left\|(\Delta^{\widetilde{K}_n,\widetilde{L}_n})^{-1}\right\|$.

As π/Γ_n is amenable, the work of Mathai-Rothenberg applies to the pair $(\widetilde{K}_n, \widetilde{L}_n)$. It follows from their proof [26, Prop. 2.6] that for all n,

$$\operatorname{Det}_{\pi/\Gamma_n} \Delta^{\widetilde{K}_n,\widetilde{L}_n} = 1$$

and $\Delta^{\widetilde{K}_n,\widetilde{L}_n}$ has a spectral gap at zero of size at least K^{-2}.

Now by Lemma 3.5,

$$\operatorname{Det}_{\pi} \Delta^{\widetilde{K},\widetilde{L}} = \lim_{n\to\infty} \operatorname{Det}_{\pi/\Gamma_n} \Delta^{\widetilde{K}_n,\widetilde{L}_n} = 1.$$

□

This finishes the proof of Theorem 0.3. As remarked earlier, in both [21] and [26], the approximating spaces are finite. The approximating Laplacians have determinant 1 and spectral gap because they are finite invertible integer matrices, and the previous proposition essentially boils down to this fact.

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