The Ihara zeta function of the infinite grid

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An Integral

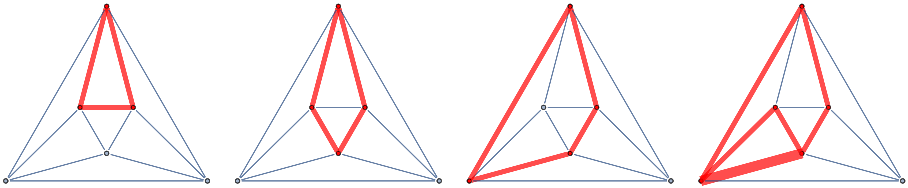
$$\mathbf{I}(k) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log\left[1 - \frac{k}{2}(\cos t + \cos s)\right] \, ds \, dt$$

The zeta function of a finite graph

X a graph.

Loop A closed path in X, up to cyclic equivalence, without backtracking.

Prime A loop which is not a power of another loop.



Primes in the octahedral graph

The zeta function of a finite graph

The zeta function of a graph X is

$$Z(u) = \prod_{\gamma \text{ prime}} \frac{1}{1 - u^{\text{length}(\gamma)}},$$

which converges for $u \in \mathbb{C}$ near 0.

Thm (Ihara '66, Hashimoto '89, Bass '92)

X a graph with v vertices, e edges. Let A be the adjacency matrix, Q be the $v \times v$ diagonal matrix with $Q_{ii} = \deg v_i - 1$, and $\Delta_u = I - uA + u^2 Q$. Then

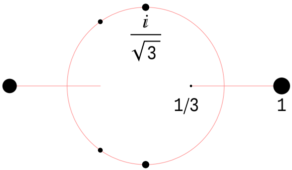
$$Z(u)^{-1} = (1 - u^2)^{e-v} \det \Delta_u$$

Example

Octahedral graph

$$\Delta_u = \begin{pmatrix} 3u^2+1 & -u & -u & -u & -u & 0 \\ -u & 3u^2+1 & -u & -u & 0 & -u \\ -u & -u & 3u^2+1 & 0 & -u & -u \\ -u & -u & 0 & 3u^2+1 & -u & -u \\ -u & 0 & -u & -u & 3u^2+1 & -u \\ 0 & -u & -u & -u & -u & 3u^2+1 \end{pmatrix}$$

$$\begin{aligned} Z(u)^{-1} &= (1-u^2)^6 \det \Delta_u \\ &= (u-1)^7(u+1)^6(3u-1)\left(3u^2+1\right)^3\left(3u^2+2u+1\right)^2 \\ &= 1 - 16u^3 - 30u^4 - 48u^5 + 16u^6 + \cdots - 3888u^{22} + 729u^{24} \\ &= (1-u^3)^{16}(1-u^4)^{30}(1-u^5)^{48}(1-u^6)^{104}\cdots. \end{aligned}$$



Riemann Hypothesis

For a $q+1$ regular graph X, put $u = q^{-s}$.

The Riemann hypothesis for X

All poles of $Z(q^{-s})$ with $0 < \Re(s) < 1$ satisfy $\Re(s) = \frac{1}{2}$.

The spectrum of A is contained in $[-q-1, q+1]$. Put

$$\mu = \max \left\{ |\lambda| \middle| \lambda \in \text{spec}(A), |\lambda| \neq q+1 \right\}$$

Since $\Delta_u = 1 - uA + qu^2$,

$$\text{poles of } Z(u) \longleftrightarrow \text{eigenvalues of } A$$

The Riemann hypothesis for X

$$\Updownarrow$$

$$\mu \leq 2\sqrt{q}$$

$$\Updownarrow$$

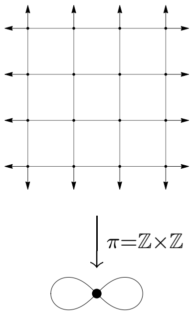
X is a Ramanujan graph

The grid zeta function

Let X be the infinite grid.
$\pi = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$ acts on X.
The zeta function is still an infinite product:

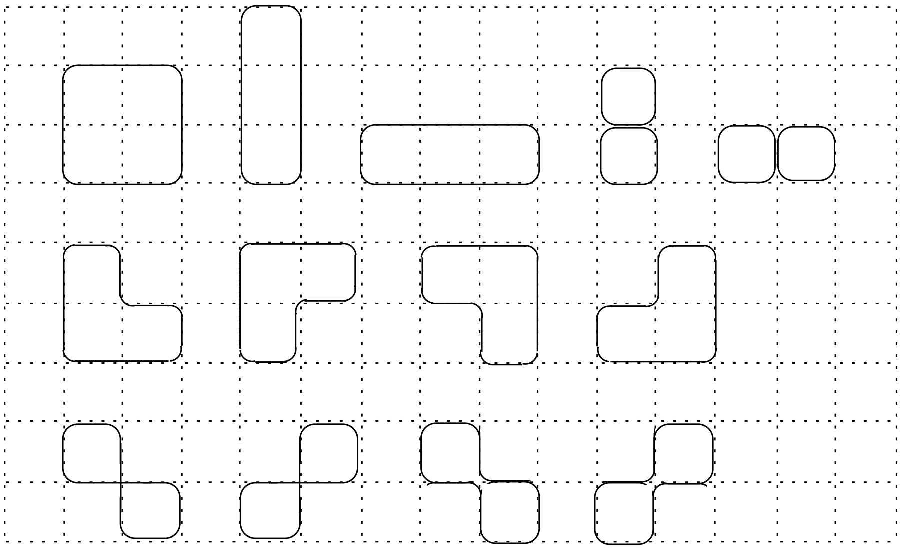
$$Z_\pi(u) = \prod_{[\gamma] \text{ prime}} \frac{1}{1 - u^{\text{length}(\gamma)}},$$

where $[\gamma]$ is an equivalence class of loops under translation by π.



$$\begin{aligned} Z_\pi(u) &= (1-u^4)^{-2}(1-u^6)^{-4}(1-u^8)^{-26}(1-u^{10})^{-152}\cdots \\ &= 1 + 2u^4 + 4u^6 + 29u^8 + 160u^{10} + 1070u^{12} + \cdots \end{aligned}$$

Prime loops of length 8



Determinant formula

On infinite graphs, the adjacency matrix becomes an adjacency *operator*. For the infinite grid,

$$\Delta_u = 1 - uA + 3u^2 : \ell^2(\mathbb{Z} \times \mathbb{Z}) \to \ell^2(\mathbb{Z} \times \mathbb{Z}).$$

There is still a determinant formula for the zeta function. Here,

$$Z_\pi(u)^{-1} = (1 - u^2) \det{}_\pi \Delta_u.$$

With $\pi = \mathbb{Z} \times \mathbb{Z}$, $\det_\pi$ is an operator determinant:

$$\det{}_\pi \Delta_u = \exp \operatorname{Tr}_\pi \log \Delta_u$$

Tr_π is the trace on the group von Neumann algebra $\mathcal{N}(\pi)$.

The zeta function integral

Put $\pi = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$. The adjacency operator on the grid is

$$A = a + a^{-1} + b + b^{-1} \in \mathbb{C}[\pi]$$

Fourier transform: $\ell^2(\mathbb{Z} \times \mathbb{Z}) \leftrightarrow L^2(S^1 \times S^1)$ with $(a, b) \leftrightarrow (e^{is}, e^{it})$.

$$\hat{A} = 2\cos s + 2\cos t$$

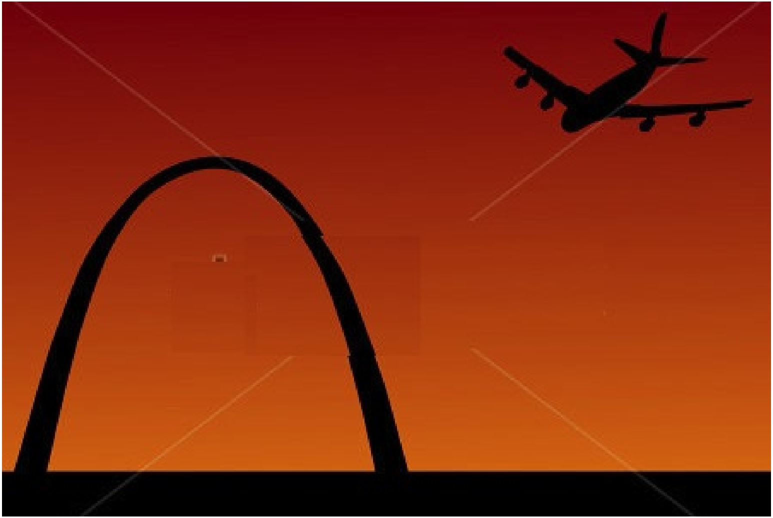
$$\hat{\Delta_u} = 1 - 2u(\cos s + \cos t) + 3u^2$$

The von Neumann trace is integration over $S^1 \times S^1$.

$$\begin{aligned}
\det{}_\pi \Delta_u &= \exp \int\int_{S^1 \times S^1} \log\left(1 + 3u^2 - 2u(\cos t + \cos s)\right)\, dsdt \\
&= (1 + 3u^2) \exp \int\int_{S^1 \times S^1} \log\left(1 - \frac{k}{2}(\cos t + \cos s)\right)\, dsdt \\
&= (1 + 3u^2) \exp \mathbf{I}(k)
\end{aligned}$$

with $k = 4u/(1 + 3u^2)$

12 years pass....



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CONVERGENCE OF ZETA FUNCTIONS OF GRAPHS

BRYAN CLAIR AND SHAHRIAR MOKHTARI-SHARGHI

ABSTRACT. The L^2-zeta function of an infinite graph Y (defined previously in a ball around zero) has an analytic extension. For a tower of finite graphs covered by Y, the normalized zeta functions of the finite graphs converge to the L^2-zeta function of Y.

INTRODUCTION

Associated to any finite graph X there is a zeta function $Z(X, u)$, $u \in \mathbb{C}$. It lefined as an infinite product but shown (in various different cases) by Ihara, shimoto, and Bass [5, 4, 1] to be a polynomial. Indeed the rationality formula

12 years pass....

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To say "growth rates" suggests a tower, or at least a sequence of covering
the main theorems in this paper are stated as bounds on given finite cover:
X:

Theorem 0.1. *Let X be a finite simplicial complex, and $\widetilde{X}$ an infinite covering with covering group Γ. Suppose that $b_q^{(2)}(\widetilde{X};\Gamma) = 0$, or equivalent there are no L^2 harmonic q-cochains on $\widetilde{X}$.*

1. *(Spectral Gap) Suppose there is a gap near 0 in the L^2 spectrum o dimension q. Then there are $C > 0$ and $M > 0$ so that for any finite cover $X' = \widetilde{X}/\Gamma'$ of X:*

$$b_q(X') \leq C\frac{[\Gamma : \Gamma']}{e^{M\text{short}(\Gamma')}}.$$

2. *(Positive Novikov-Shubin Invariant) If $\widetilde{X}$ has Novikov-Shubin invariar 0, then for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ so that for any finite regula*

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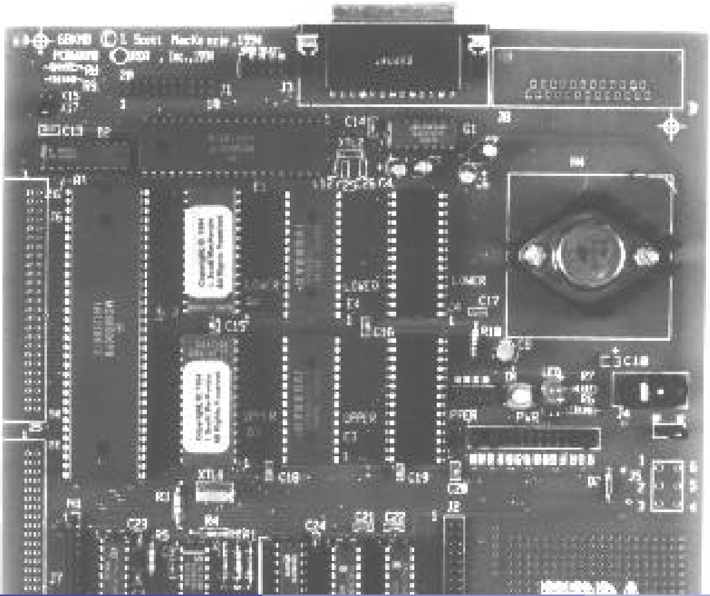

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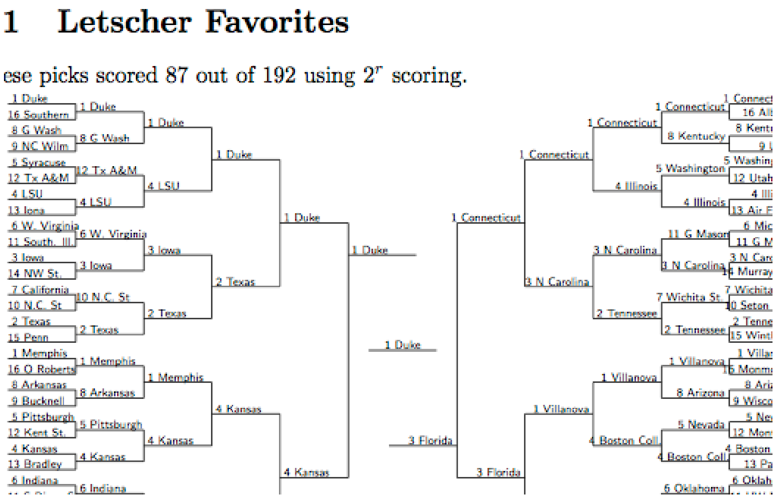

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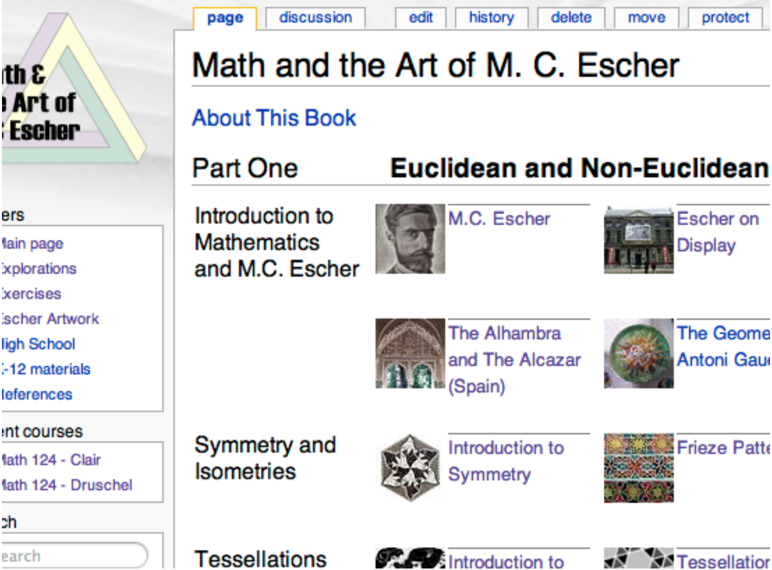

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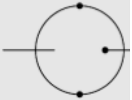
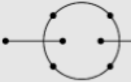
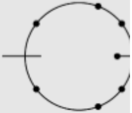
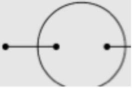
Table 1
Some graphs with $n = 1$

#	Graph	q	$\chi(X)$	$r(u)$	Branchpoints
1		3	−1	$\frac{1-2u+3u^2}{2u}$	
2		4	−3	$\frac{1-9u^2+16u^4}{8u^2}$	
3		2	−2	$\frac{1-u+u^2-3u^3+2u^4-4u^5+8u^6}{4u^3}$	
4		3	−2	$\frac{1+3u^2}{4u}$	

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THIS IS INCORRECT.

1 Cellular Covers

Let X be a locally finite metric space (i.e. metric balls are finite uniformly?)).

For $Y \subset X$, define the ℓ-neighborhood of Y:

$$N_\ell(Y) = \{x \in X | d(x, Y) \leq \ell\}$$

The diameter of Y is $\max_{y_1, y_2 \in Y} d(y_1, y_2)$

Suppose we have the following data:

- A control "size" $K > 0$.
- A "dimension" $n \geq 0$.

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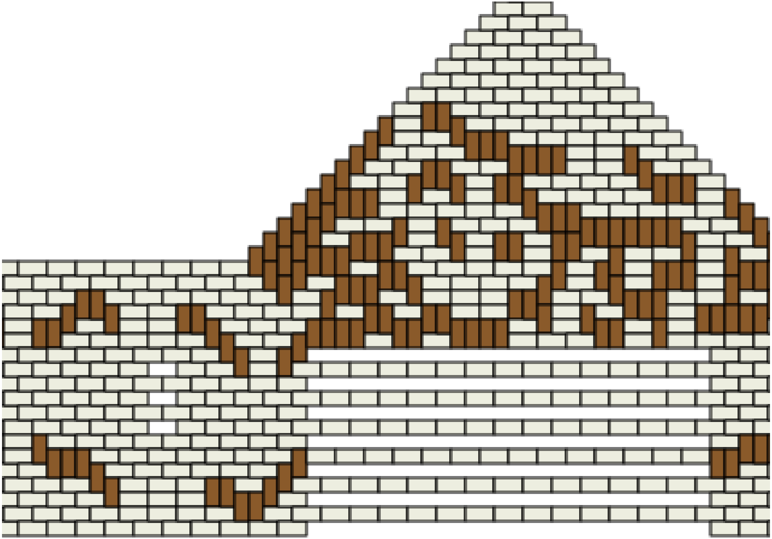
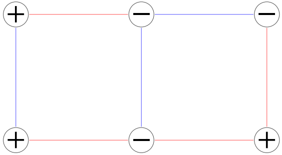


The Ising model

X a finite graph. A *state* σ is an assigment of spins ± 1 to each vertex. The interaction energy of a state is

$$\varepsilon(\sigma) = -\sum_{v_i \sim v_j} \sigma(v_i)\sigma(v_j) = \#(\text{unlike spins}) - \#(\text{like spins})$$



has energy $4 - 3 = 1$.

The Gibbs distribution:

$$\text{Prob}(\sigma) \sim e^{-\beta\varepsilon(\sigma)}$$

where $\beta = 1/(kT)$, k is Boltzmann's constant, T is temperature.

The Ising model

Need to compute the *partition function*

$$Z(\beta) = \sum_{\sigma} e^{-\beta\varepsilon(\sigma)}$$

$$\text{Prob}(\sigma) \sim \frac{e^{-\beta\varepsilon(\sigma)}}{Z(\beta)}$$

- $T \to \infty$: $\beta \to 0$, $Z(\beta) \to 2^v$, all states equally likely.
- $T \to 0$: $\beta \to \infty$, only lowest energy states have positive probability. The system is magnetized.

The Ising model

Let X_N be the square $N \times N$ grid, and $Z(X_N, \beta)$ the partition function.

Thm (Kasteleyn '63)

$$\lim_{N \to \infty} Z(X_N, \beta)^{2/N} = 4 \cosh^2(\beta) \exp \mathbf{I}(k)$$

with $k = 2 \sinh \beta / \cosh^2 \beta$.

Integration

$$\mathbf{I}(k) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log\left[1-\frac{k}{2}(\cos t+\cos s)\right]\,ds\,dt$$
$$= \frac{2}{\pi^2}\int_0^{\pi/2}\int_0^{\pi}\log\left[1-k\cos(\tau)\cos(\omega)\right]d\tau d\omega$$
$$= \frac{2}{\pi}\int_0^{\pi/2}\log\frac{1}{2}\left[1+\sqrt{1-k^2\sin^2(\omega)}\right]d\omega$$

Take the derivative w.r.t. k

$$\mathbf{I}'(k) = \frac{1}{k}\left(1-\frac{2}{\pi}\int_0^{\pi/2}\frac{d\omega}{\sqrt{1-k^2\sin^2(\omega)}}\right)$$
$$= \frac{1}{k}\left(1-\frac{2}{\pi}\mathbf{K}(k)\right)$$

$\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

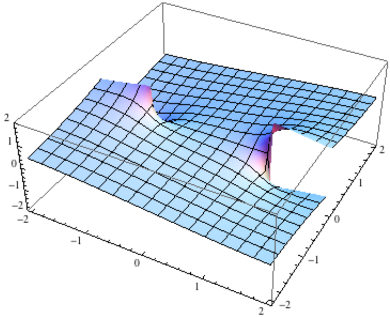
The story thus far...

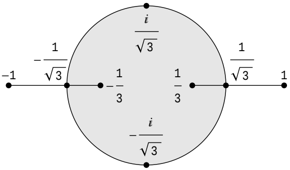
$$Z_\pi(u)^{-1} = (1-u^2)(1+3u^2)\exp \mathbf{I}(k)$$

$$k = \frac{4u}{(1+3u^2)}$$

$$\mathbf{I}'(k) = \frac{1}{k}\left(1 - \frac{2}{\pi}\mathbf{K}(k)\right)$$

$\Im \mathbf{K}(k)$





Theta functions

- $\tau \in \mathbb{H}$, the upper half plane
- The "nome" $q = e^{\pi i \tau}$

$$\theta_2 = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} = 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2$$

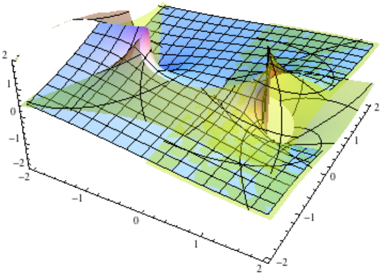
$$\theta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2$$

$$\theta_4 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2$$

The squares of the theta functions are modular forms of weight 1.

Uniformization

$$\mathbf{K} = \frac{\pi}{2}\theta_3^2 \qquad k = \frac{\theta_2^2}{\theta_3^2}$$



A plot of $\tau \to (k(\tau), \mathbf{K}(\tau))$.

Uniformization

$$\mathbf{I}' = \frac{1}{k}\left(1 - \frac{2}{\pi}\mathbf{K}(k)\right) = \frac{\theta_3^2(1-\theta_3^2)}{\theta_2^2}$$

- $\mathbf{I}'$ is analytic as a funtion of $\tau \in \mathbb{H}$
- $\mathbf{I}$ is analytic as a funtion of $\tau \in \mathbb{H}$
- $\tau \to (k(\tau), \mathbf{I}(\tau))$ takes all values of $\mathbf{I}$

Uniformization

Let S be the Riemann surface

$$S = \left\{ (u, \tau) \in \mathbb{C} \times \mathbb{H} \;\middle|\; k(u) = k(\tau) \right\}$$

$$\frac{4u}{1+3u^2} = \frac{\theta_2^2(\tau)}{\theta_3^2(\tau)}$$

Thm

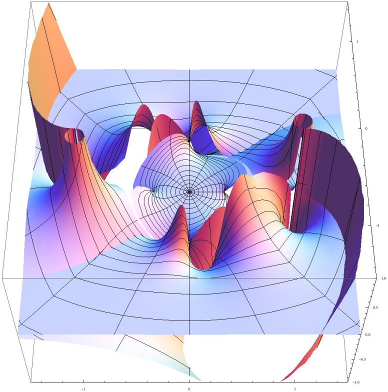
The zeta function of the grid is defined on S by

$$Z_\pi^{-1} = (1-u^2)(1+3u^2)\exp \mathbf{I}(k)$$

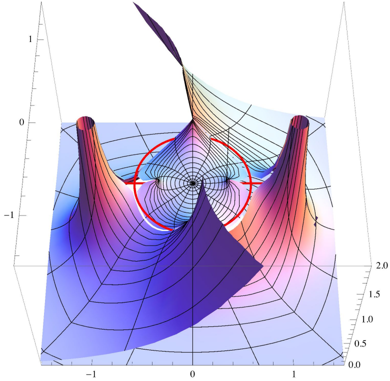
It has isolated singularities which lie over the set

$$u = \{0, \pm\frac{1}{3}, \pm\frac{1}{\sqrt{3}}, \pm\frac{i}{\sqrt{3}}, \pm 1\}$$

The grid zeta function



$\Im Z_\pi$



$|Z_\pi|$