Due Monday, April 14, at the start of class. Name: ____________________

Math 320 – Take Home Quiz 4

This quiz should take you approximately 25 minutes. You may use your calculator, your book, and your notes, but do not work together and do not get help. You are allowed to use Matlab/Octave, but it is not recommended.

1. Use the composite trapezoid rule with $n = 5$ subdivisions to approximate $\int_1^{1.5} \frac{1}{x} dx$.

> **Solution:**
> $$\int_1^{1.5} \frac{1}{x} dx \approx \frac{0.1}{2}\left(\frac{1}{1} + 2\frac{1}{1.1} + 2\frac{1}{1.2} + 2\frac{1}{1.3} + 2\frac{1}{1.4} + \frac{1}{1.5}\right) \approx 0.4059274$$

2. Use the error term $-(b-a)\frac{h^2}{12}f''(\xi)$ from the composite trapezoid rule to bound the error of the approximation in question 1.

> **Solution:** Here $f(x) = 1/x$ and $f''(x) = 2/x^3$. The maximum value of f'' on the interval $[1, 1.5]$ is at $x = 1$, when $f''(x) = 2$. The error for the approximation is less than $(.5) * (.1)^2 * 2/12 \approx .00083$. Note that the actual error is $0.4059274 - \ln(5) \approx .00046$.

3. Show that the midpoint rule has degree of precision equal to one.

Solution: Degree one means the midpoint rule is exact for degree one polynomials. So, let $mx + h$ be a line. The midpoint rule gives

$$\int_a^b mx + h dx \approx (b-a)\left(m\left(\frac{b+a}{2}\right) + h\right) = m\frac{b^2 - a^2}{2} + h(b-a)$$

while the fundamental theorem computes the integral as

$$\int_a^b mx + h dx = m\frac{x^2}{2} + hx\Big|_a^b = m\frac{b^2 - a^2}{2} + h(b-a).$$

The midpoint rule gives the exact value of the integral.

For completeness, one should show the midpoint rule is *not* exact for degree two polynomials. One example:

$$\frac{1}{3} = \int_0^1 x^2 \neq 1 \cdot \left(\frac{1}{2}\right)^2.$$

4. The composite trapezoid rule applied to $\int_0^4 \sin(\frac{x^2}{4})dx$ gives the results below for $h = .4, .2, .1$. Use two steps of extrapolation to improve the results (this is Romberg integration).

h	Approximation
.4	1.591849
.2	1.605178
.1	1.608462

Solution:

h	Approximation	$k = 2$	$k = 4$
.4	1.591849		
.2	1.605178	1.609621	
.1	1.608462	1.609557	1.609553

The extrapolated value is 1.609553.

(Which is actually the correct value of the integral to all given decimal places).

5. (a) Using the composite Trapezoid Rule to approximate $\int_1^5 \frac{e^x}{x} dx$ with $h = .2$ gives 38.37 with an error of 0.08.

Approximately what error would result from using the composite Trapezoid rule with $h = .02$?

> **Solution:** Multiplying h by .1 multiplies the error by $(0.1)^2 = .01$, so the error will be approximately .0008.

(b) Using the composite Simpson's Rule to approximate $\int_1^5 \frac{e^x}{x} dx$ with $h = .2$ gives 38.290170 with an error of 1.27×10^{-5}.

Approximately what error would result from using composite Simpson's rule with $h = .02$?

> **Solution:** Multiplying h by .1 multiplies the error by $(0.1)^4 = 10^{-4}$, so the error will be approximately 1.27×10^{-9}.