

## Math 320 – Take Home Quiz 3

This quiz should take you approximately 25 minutes. You may use your calculator, your book, and your notes, but do not work together and do not get help. You are allowed to use Matlab/Octave, but it is not recommended.

- (10) 1. On the left are five interpolating polynomials with nodes  $\{1, 2, 3\}$ , written in Lagrange form. On the right are the same five polynomials, written in Newton form. Match them!

Lagrange Forms

Newton Forms

$$L_1 = 4(2-x)(3-x) + 3(x-1)(3-x) + (x-2)(x-1)$$

$$N_1 = (2(x-2) + 1)(x-1) + 2$$

$$L_2 = (2-x)(3-x) + 5(x-1)(3-x) + 4(x-2)(x-1)$$

$$N_2 = (2(x-2) - 5)(x-1) + 8$$

$$L_3 = (2-x)(3-x) + 3(x-1)(3-x) + 4(x-2)(x-1)$$

$$N_3 = (1 - 2(x-2))(x-1) + 4$$

$$L_4 = 2(2-x)(3-x) + 5(x-1)(3-x) + (x-2)(x-1)$$

$$N_4 = 3(x-1) + 2$$

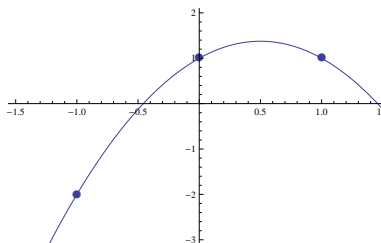
$$L_5 = 2(2-x)(3-x) + 5(x-1)(3-x) + 4(x-2)(x-1)$$

$$N_5 = (x-1)^2 + 4$$

**Solution:**  $L_1 = N_2$ ;  $L_2 = N_4$ ;  $L_3 = N_1$ ;  $L_4 = N_3$ ;  $L_5 = N_5$ ;

- (10) 2. Make a large, accurate sketch with the points  $(-1, -2)$ ,  $(0, 1)$ , and  $(1, 1)$  shown. Calculate the degree 2 interpolating polynomial for these points and sketch it, too.

**Solution:**  $P_2(x) = -\frac{3}{2}x^2 + \frac{3}{2}x + 1$ .



- (10) 3. Compute the forward difference table  $\Delta^k f$  and explain why the polynomial interpolating the data has degree 3.

$x$	-2	-1	0	1	2	3
$f(x)$	-27	-8	1	6	13	28

**Solution:**

	-27	-8	1	6	13	28
	19	9	5	7	15	
		-10	-4	2	8	
		6	6	6		
			0	0		
				0		

Since the forward differences  $\Delta^k f$  are zero for  $k > 3$ , the polynomial is cubic.

- (10) 4. A table of cosines has values spaced by .01 from  $x = 0$  to  $x = 1.57 (\approx \pi/2)$ . Find a good bound on the error for linear interpolation using this table.

**Solution:** Let  $f(x) = \cos(x)$ , so  $f''(x) = -\cos(x)$ , and  $|f''| \leq 1$ .  
 For  $x \in [x_i, x_{i+1}]$ ,  $|(x - x_i)(x - x_{i+1})| \leq \left(\frac{.01}{2}\right) \left(\frac{.01}{2}\right)$ .  
 So, if  $P(x)$  is linear interpolation between  $x_i$  and  $x_{i+1}$ , there is some  $\xi$  with

$$|\cos(x) - P(x)| = \left| \frac{f''(\xi)}{2!} (x - x_i)(x - x_{i+1}) \right| \leq \frac{1}{2} \left( \frac{.01}{2} \right)^2 = 1.25 \times 10^{-5}.$$

(10) 5. Let  $S$  be the natural cubic spline interpolant for a function  $f$  on nodes  $x_0 < x_1 < \cdots < x_n$ .

(a) If  $\ell(x) = mx + b$  is a linear function, show that  $S + \ell$  is the natural cubic spline interpolant for  $f + \ell$ .

**Solution:**

- $S + \ell$  is still piecewise cubic.
- $(S + \ell)(x_i) = S(x_i) + \ell(x_i) = f(x_i) + \ell(x_i) = (f + \ell)(x_i)$  so that  $S + \ell$  interpolates  $f + \ell$  at the nodes.
- Since  $(S + \ell)' = S' + m$ , and  $S'$  is continuous, so is  $(S + \ell)'$ .
- Since  $(S + \ell)'' = S''$ , and  $S''$  is continuous, so is  $(S + \ell)''$ .
- $S$  is natural, so  $(S + \ell)''(x_0) = S''(x_0) = 0$ , and  $(S + \ell)''(x_n) = S''(x_n) = 0$ , so  $S + \ell$  is natural.

(b) If  $q(x) = ax^2 + bx + c$  is a quadratic function, explain why  $S + q$  is not the natural cubic spline interpolant for  $f + q$ .

**Solution:** Because  $(S + q)''(x_0) = S''(x_0) + 2a = 2a \neq 0$ , so  $S + q$  is not natural. (It *is* a cubic spline interpolant, though!)