Due Wednesday February 5, in class   Name: ____________________

Math 320 – Take Home Quiz 1

This quiz should take you approximately 25 minutes. You may use your calculator, your book, and your notes, but do not work together and do not get help. You are allowed to use Matlab/Octave, but it is not recommended.

(10) 1. (a) Give an example of a number p and an approximation p^* so that the absolute error is bigger than 1 but the relative error is less than .001

Solution: $p = 100000$, $p^* = 100002$. Absolute error is 2, relative error is .00002.

(b) Give an example of a number p and an approximation p^* so that the absolute error is less than .001 but the relative error is larger than 1.

Solution: $p = .00001$, $p^* = .00003$. Absolute error is .00002, relative error is 2.

(10) 2. Compute $2^{-1} + 2^{-52}$ to 20 significant digits.

Solution: 0.50000000000000022204

(10) 3. The function $f(x) = 3x - e^x$ has a root in the interval $[1, 2]$. Show the first three steps of the bisection method to find this root. How many steps will it take to find the root to within .001?

Solution: $f(1) \approx .28 > 0$, $f(2) \approx -1.4 < 0$.

1. Midpoint is 1.5. $f(1.5) \approx .018 > 0$, so use $[1.5, 2]$
2. Midpoint is 1.75. $f(1.75) \approx -.5 < 0$, so use $[1.5, 1.75]$
3. Midpoint is 1.625. $f(1.625) \approx -.2 < 0$, so use $[1.5, 1.625]$

At step n, the width of the interval is 2^{-n}. For $2^{-n} < .001$, $n > 9.97$, so it takes 10 steps.

(10) 4. Consider the series

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \frac{1}{8 \times 9 \times 10} + \cdots$$

Approximately how many terms of this series do you need in order to compute $\frac{\pi}{4}$ to 150 decimal places?

Solution: This is an alternating series, so the error is bounded by the first unused term. The n^{th} term (up to sign) is $\frac{1}{2n(2n+1)(2n+2)}$ which is a bit smaller than $\frac{1}{(2n)^3}$. For 150 decimal places, we need

$$(2n)^{-3} < 10^{-150}$$

or

$$2n > 10^{50}.$$

You need approximately $.5 \times 10^{50}$ terms.

(10) 5. Show that $g(x) = 0.5\cos(x)$ has a fixed point $p \in [0, 1]$, and that fixed point iteration $x_n = g(x_{n-1})$ converges to p for any $x_0 \in [0, 1]$.

Solution: For $x \in [0, 1]$, $\cos(1) \approx .54 < \cos(x) \leq 1$, so $.27 < g(x) \leq .5$. In other words, $g([0, 1]) \subset [0, 1]$. Since g is continuous, g has a fixed point in the interval $[0, 1]$.

Now $|g'(x)| = |-0.5\sin(x)| \leq 0.5$ for all x. So, with $k = 0.5$, $g'(x) \leq k < 1$ for $x \in [0, 1]$, which shows that fixed point iteration will converge to the unique fixed point p.

Give a bound for the absolute error in the approximation $x_{20} \approx p$ after 20 steps of fixed point iteration. Your answer should be valid for any $x_0 \in [0, 1]$.

Solution: From the convergence theorem, with $b - a = 1$ and $k = 0.5$, the error is bounded as $|x_n - p| \leq 0.5^n$. With $n = 20$, the absolute error is bounded by 9.54×10^{-7}.

This bound is not great - in practice this error is actually achieved after about 9 steps, and 20 steps does much better.