Monday, May 5

Math 320 – Review Questions

1. Given an ODE problem $y' = f(t, y)$, $a \le t \le b$, $y(a) = y_0$, let $h = \frac{b-a}{N}$ and $t_i = a + ih$. The Backwards Euler Method for solving puts $w_0 = y_0$ and computes:

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}).$$

This method is implicit - the variable w_{i+1} appears on both sides of the equation, and you need to solve for w_{i+1} to take the step.

Use this method with $N = 5$ to solve $y' = -10y$, $0 \le t \le 1$, $y_0 = 1$, and compute the approximation to $y(1)$.

Repeat with $N = 10$.

Solution: The Backwards Euler Method becomes

$$w_{i+1} = w_i - 10hw_{i+1}$$

or

$$w_{i+1} = \frac{w_i}{1 + 10h}.$$

Then when $N = 5$, $h = 0.2$ and each step gives $w_{i+1} = \frac{1}{3}w_i$. Since $w_0 = 1$, $w_i = 3^{-i}$. $y(1) \approx 3^{-5} = 0.0041$.

When $N = 10$, $h = 0.1$ and each step gives $w_{i+1} = \frac{1}{2}w_i$. Since $w_0 = 1$, $w_i = 2^{-i}$. $y(1) \approx 2^{-10} = 0.00097$.

The exact solution is $y(t) = e^{-10t}$ and $y(1) = 0.000045$.

2. Consider the differential equation $y' = ty^{\alpha}$, where $0 \le t \le 1$, $y_0 = 1$, and $\alpha > 0$ is a constant.

   (a) Find a Lipshitz constant (depending on α) for f in the y variable on the domain

   $$D = \{(t, y) | 0 \le t \le 1 \quad \text{and} \quad 1 \le y \le 2\}.$$

   (b) If you use Eulers method with $n = 100$, will you get a more accurate solution for the $\alpha = 1$ problem or the $\alpha = 2$ problem?

Solution: Compute $\frac{\partial f}{\partial y} = t\alpha y^{\alpha-1}$. On D, we have $\left|\frac{\partial f}{\partial y}\right| \le \alpha 2^{\alpha-1} = L$.

When $\alpha = 1$, $L = 1$. When $\alpha = 2$, $L = 4$. Since smaller Lipshitz constants result in less error in Euler's method, we expect the solution to be more accurate when $\alpha = 1$. (And it is, too: The error is 0.0109 for $\alpha = 1$ and 0.0382 when $\alpha = 2$)

3. Computing the integral $\int_0^1 e^{-x^2}dx$ with Simpson's rule and $h = 0.25$ gives 0.746855. Using $h = 0.125$ gives 0.746826.

   Use extrapolation to compute a more accurate result.

> **Solution:** We expect that cutting h in half from 0.25 to 0.125 will cut the error in Simpson's rule by 1/16. If the error when $h = 0.125$ is ϵ, then
>
> $$15\epsilon \approx 0.746826 - 0.746855,$$
>
> so $\epsilon \approx -0.000002$, so extrapolation gives $0.746826 - 0.000002 = 0.746824$ as a more accurate result.

4. Show that the midpoint rule for quadrature has degree of precision equal to one.

> **Solution:** The midpoint approximation is
>
> $$\int_a^b f(x)dx \approx (b-a) \cdot f(\frac{a+b}{2}).$$
>
> When $f(x) = 1$, the midpoint approximation gives $(b-a) \cdot 1$, and $\int_a^b 1dx = b - a$. When $f(x) = x$, the midpoint approximation gives
>
> $$(b-a) \cdot \frac{(a+b)}{2} = \frac{1}{2}(b^2 - a^2) = \frac{1}{2}x^2\Big|_a^b = \int_a^b xdx.$$
>
> When $f(x)$ is degree 2, the midpoint approximation is no longer exact. For example, $\int_0^1 x^2dx = 1/3$, but the midpoint approximation is $(1/2)^2 = 1/4$.

5. (a) Perform one step of Euler's method on the differential equation $y' = y + t$ with $y_0 = 1$, $0 \le t \le 0.5$ and $h = 0.5$.
   (b) Repeat with the modified Euler's method.
   (c) Use part (b) to estimate the error in your result from part (a).

> **Solution:**
>
> (a) $w_0 = 1$, and $m = 1 + 0 = 1$, so $w_1 = w_0 + .5 * 1 = 1.5$.
>
> (b) With $m_1 = 1$ from part a, $m_2 = f(.5, 1.5) = 2$, and $m = (m_1 + m_2)/2 = 1.5$ so $\tilde{w_1} = 1 + .5 * 1.5 = 1.75$.
>
> (c) Since modified Euler's method is higher order, we estimate the error in part (a) as $|1.5 - 1.75| = 0.25$.