SLU Missouri Collegiate Math Team – Qualifying Exam 2019
April 8, 2019

Instructions:

- Please work on these problems independently without help from other students or outside sources.
- Explain your ideas as clearly as possible.
- Don't worry about solving everything, just do what you can.
- Please submit your work electronically (photo/scan/typed) by 11:59pm on Sunday, March 24th.

1. Consider the ellipse $x^2 + 2y^2 = 1$. Let R_a be the rectangle defined by the opposite corners $(0, 0)$ and $P = (a, \sqrt{\frac{1}{2}(1 - a^2)})$. For what $a > 0$ is the area of R_a maximal?

Solution: The area of the rectangle is $A = xy$, where (x, y) corresponds to the coordinates of the point P. Thus,

$$\frac{dA}{dx} = \frac{d}{dx}[xy] = y + x\frac{dy}{dx}.$$

Using implicit differentiation, notice that

$$2x + 4y\frac{dy}{dx} = 0,$$

so that $\frac{dy}{dx} = -\frac{x}{2y}$. Hence,

$$\frac{dA}{dx} = y + x\frac{dy}{dx} = y - \frac{x^2}{2y} = 0$$

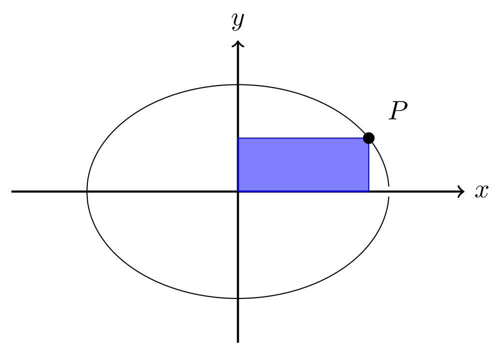
if and only if

$$x^2 = 2y^2.$$

Substituting this expression for $2y^2$ into the equation of the ellipse leads to $x^2 + x^2 = 1$ or $x = \frac{1}{\sqrt{2}}$. It follows that $y = \frac{1}{2\sqrt{2}}$. Notice that as x increases, y decreases, so the sign of $\frac{dA}{dx}$ will go from positive to negative at this critical point and the area must be maximal when

$$a = \frac{1}{\sqrt{2}}.$$

□



2. Calculate $\frac{d^{2019}}{dx^{2019}}\left[\frac{1}{1 - x^3}\right]\Big|_{x=0}$.

Solution: For $|x| < 1$, the function of interest can be written as a geometric series:

$$\frac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + \cdots.$$

This is also the Taylor series for the function at $x = 0$ and thus obeys the general form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x - 0)^n.$$

Therefore, the 2019th derivative at $x = 0$ is determined from the coefficient of x^{2019}, leading to

$$\frac{f^{(2019)}(0)}{2019!} = 1 \qquad \text{or} \qquad f^{(2019)}(0) = 2019!.$$

□

3. Let A be the 2×2 matrix shown below. Describe all 2×2 matrices B that satisfy $AB = BA$.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Solution: To make the problem more concrete, define B as follows:

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now compute AB and BA to get:

$$AB = \begin{pmatrix} a+c & b+d \\ -a+c & -b+d \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} a-b & a+b \\ c-d & c+d \end{pmatrix}.$$

Equating the entries of the two products leads to four equations:

$$a + c = a - b \qquad b + d = a + b \qquad -a + c = c - d \qquad -b + d = c + d.$$

After canceling like terms from each side of the equation, only two distinct equations remain: $c = -b$ and $a = d$. It follows that $AB = BA$ if and only if B has the form

$$B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

□

4. Recall from the 2017 Missouri Collegiate Mathematics Competition that an *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, 6,210,001,000 is an autobiographical number. Prove that an autobiographical number cannot have a 7 in any digit.

Solution: The proof will make use of an interesting fact surrounding autobiographical numbers. Since each digit of an autobiographical number describes the number of digits with a certain value, the sum of the digits must equal the number of digits in the number. For example, 1210 is an autobiographical number (1 zero, 2 ones, 1 two) and has $1 + 2 + 1 + 0 = 4$ digits. Similarly, 6,210,001,000 has 10 digits.

Assume that there is an autobiographical number with a seven digit.

(1) *The number cannot have two digits equal to 7.* If two digits equal 7, then the sum of the digits would exceed 10.

(2) *The first digit must be 7.* If the seven occurs anywhere other than the first digit, then the sum of the digits will exceed 10. For example, if the second digit is 7 then there must be 7 ones and the sum of the digits will be no less than 14.

(3) *The second digit is at least 2.* Since there is only one 7 in the number, the eighth digit must be 1 (indicating the number of 7's). Since there is at least one 1, the second digit cannot be zero. If the second digit is 1 that would indicate one 1, but the eighth digit is also 1, so the number would not be autobiographical. This shows that the second digit is at least 2.

(4) *The sum of the digits is at least 11.* Based on the previous observations, there must be at least two 1's and one 7. Given that the second digit is not zero or 1, the sum of all of the digits must include another number that is no smaller than 2. This leads to a sum of at least $7 + 1 + 1 + 2 = 11 > 10$.

It follows that there is no autobiographical number with a seven among its digits. However, if we switch to base 11 and introduce a new digit $a = 10$, then autobiographical numbers can have 11 digits and 72 100 001 000 becomes autobiographical. (The last zero tells us we have 0 a's in the number.) □

5. A local fast-food restaurant is planning to give away one of four possible transformer toys with each kid's meal purchased. The toy placed in each kid's meal is randomly selected and each toy is equally likely. A local collector wants to acquire one of each toy and is willing to buy up to ten kid's meals to accomplish this goal.

   a. What is the probability that the collector will acquire all four toys by purchasing just four kid's meals?

      Solution: With four toys and four kid's meals, there are 4^4 ways to assign toys to the different meals. Each of these can be represented by a four digit number that uses only the digits 1, 2, 3, and 4. Only the numbers with all four digits correspond to situations where the collector receives all four toys, e.g., 1234 or 4321. There are 4! ways to rearrange the digits of 1234, so there are 4! ways to assign all four toys to the four boxes, leading to a probability of

      $$\text{Probability} = \frac{4!}{4^4} = \frac{24}{256} = \frac{3}{32}.$$

      □

   b. What is the probability that the collector will acquire all four toys by purchasing ten kid's meals?

      Solution: This solution was provided by Dr. Bryan Clair. Let A_i be the event "toy i is not received in any of the 10 meals", for $i = 1, 2, 3, 4$.
      Then by the inclusion-exclusion principle (and noting that $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$)

      $$\begin{aligned} P(\cup_{i=1}^4 A_i) &= \sum_i P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) & (1) \\ &= 4\left(\frac{3}{4}\right)^{10} - 6\left(\frac{2}{4}\right)^{10} + 4\left(\frac{1}{4}\right)^{10} & (2) \\ &= \frac{4 \cdot 3^{10} - 6 \cdot 2^{10} + 4}{4^{10}} = \frac{230056}{4^{10}} \approx 0.219 & (3) \end{aligned}$$

      So the probability of *not* getting all four toys is $\frac{230056}{4^{10}}$, which means the probability of getting all four toys is $\frac{818520}{4^{10}} \approx 0.78060\ldots$ □

      Alternate Solution: Rather than work out the specific case of n kid's meals, it will be convenient to introduce a more general approach to the problem. Let n represent the number of kid's meals purchased. The following notation will be useful.

      - A_n will represent the number of ways to assign only one distinct toy to the n meals.
      - B_n will represent the number of ways to assign two distinct toys to the n meals.
      - C_n will represent the number of ways to assign three distinct toys to the n meals.
      - D_n will represent the number of ways to assign four distinct toys to the n meals.

      Notice that $A_n = 4$ for all n, because A_n counts the ways that all of the meals have the same toy. This corresponds to a [trivial] recursive formula, $A_{n+1} = A_n$. However, there are more interesting recursive formulas for the other quantities. Consider the quantity B_{n+1}. For each instance counted by B_n there are two ways to add a meal without increasing the number of distinct toys. Similarly, for each instance counted by A_n, there are three ways to add a meal and end up with two distinct toys. This leads to the recursive formula

      $$B_{n+1} = 2B_n + 3A_n.$$

      Similar reasoning leads to recursive formulas for C_{n+1} and D_{n+1},

      $$C_{n+1} = 3C_n + 2B_n \qquad \text{and} \qquad D_{n+1} = 4D_n + C_n.$$

      Combining the recursive formulas with the initial values $A_1 = 4$ and $B_1 = C_1 = D_1 = 0$, it is possible to calculate the value of each variable for any n.

n	A_n	B_n	C_n	D_n
1	4	0	0	0
2	4	12	0	0
3	4	36	24	0
4	4	84	144	24
5	4	180	600	240
6	4	372	2160	1560
7	4	756	7224	8400
8	4	1524	23184	40824
9	4	3060	72600	186480
10	4	6132	223920	818520

The probability that the collector obtains all 4 toys with 10 kid's meals is thus

$$\text{Probability} = \frac{818,520}{4^{10}} = \frac{818,520}{1,048,576} \approx 0.78060...$$

$\square$