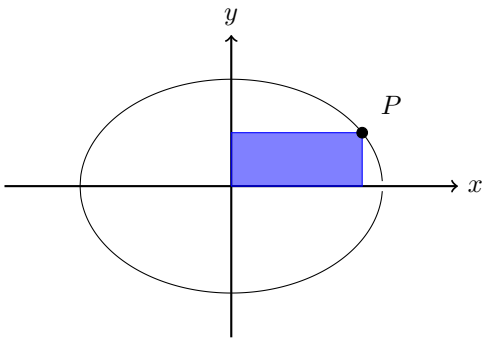
SLU Missouri Collegiate Math Team – Qualifying Exam 2019
March 17, 2019

Instructions:

- Please work on these problems independently without help from other students or outside sources.
- Explain your ideas as clearly as possible.
- Don't worry about solving everything, just do what you can.
- Please submit your work electronically (photo/scan/typed) by 11:59pm on Sunday, March 24th.

1. Consider the ellipse $x^2 + 2y^2 = 1$. Let R_a be the rectangle defined by the opposite corners $(0, 0)$ and $P = (a, \sqrt{\frac{1}{2}(1 - a^2)})$. For what $a > 0$ is the area of R_a maximal?



2. Calculate $\dfrac{d^{2019}}{dx^{2019}}\left[\dfrac{1}{1-x^3}\right]\Big|_{x=0}$.

3. Let A be the 2×2 matrix shown below. Describe all 2×2 matrices B that satisfy $AB = BA$.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

4. Recall from the 2017 Missouri Collegiate Mathematics Competition that an *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, 6,210,001,000 is an autobiographical number. Prove that an autobiographical number cannot have a 7 in any digit.

5. A local fast-food restaurant is planning to give away one of four possible transformer toys with each kid's meal purchased. The toy placed in each kid's meal is randomly selected and each toy is equally likely. A local collector wants to acquire one of each toy and is willing to buy up to ten kid's meals to accomplish this goal.

   a. What is the probability that the collector will acquire all four toys by purchasing just four kid's meals?

   b. What is the probability that the collector will acquire all four toys by purchasing ten kid's meals?