SLU Missouri Collegiate Math Team – Qualifying Exam 2018
March 26, 2018

Instructions:

- Please work on these problems independently.
- Explain your ideas as clearly as possible.
- Don't worry about solving everything, just do what you can.
- Please submit your work electronically (photo/scan/typed) by 5pm on Friday, March 30th.

The Problems:

1. For $x > 0$ let R_x be the rectangle with its lower-left corner at the origin and its upper-right corner at the point (x, e^{-x}). What is the maximum possible area of R_x, $x > 0$?

   Solution: The area of R_x is given by $f(x) = xe^{-x}$ and $f'(x) = (1-x)e^{-x}$. Notice that $f'(x) = 0$ only when $x = 1$, where the sign changes from positive to negative. This tells us that a local minimum occurs at $x = 1$ where $f(1) = e^{-1}$. Since $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$, it follows that the maximum possible area is $f(1) = e^{-1}$. □

2. Compute $\displaystyle\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx$.

   Solution: Subsitute $u = x^{\frac{1}{3}}$ so that $du = \frac{1}{3}x^{-\frac{2}{3}}$ and $dx = 3x^{\frac{2}{3}}\, du$. Notice that $x^{\frac{2}{3}} = u^2$, so

$$\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx = \int_0^1 \frac{3u^2}{1+u}\, du.$$

   Next, observe that

$$\frac{3u^2}{1+u} = \frac{3(u^2+2u+1) - 6u - 3}{u+1} = 3(u+1) - \frac{6(u+1) - 3}{u+1} = 3(u+1) - 6 + \frac{3}{u+1} = 3u - 3 + \frac{3}{u+1},$$

   so that

$$\int_0^1 \frac{1}{1 + x^{\frac{1}{3}}}\, dx = \int_0^1 3u - 3 + \frac{3}{u+1}\, du = \frac{3}{2}u^2 - 3u + 3\ln(1+u) = 3\ln 2 - \frac{3}{2}.$$

□

3. A *bitstring* of length n (n a positive integer) is a sequence of n ones and zeros, e.g., 0010 is a bitstring of length 4.

   a. How many bitstrings of length 10 contain three or more 1's?

   Solution: We need to enumerate the number of bit strings with three 1's, four 1's, five 1's, and so on. If there are three 1's we simply need to decide where to put them and we have exactly $\binom{10}{3}$ ways to do that. So the number of bitstrings of length 10 with three or more 1's must be

$$\binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \cdots + \binom{10}{10}$$

   which can be simplified using $\binom{N}{k-1} + \binom{N}{k} = \binom{N+1}{k}$ to

$$\binom{11}{4} + \binom{11}{6} + \binom{11}{8} + \binom{11}{10} = 330 + 462 + 165 + 11 = 968.$$

□

   b. How many bitstrings of length 10 contain at least three consecutive 1's?

   Solution: This is somewhat similar to previously seen problems with recursions. Let S_n represent the number of bitstrings of length n with three consecutive 1's. Let T_n represent the number of bitstrings of length n that do not contain three consecutive 1's, but end with 11. Similarly, let U_n represent bitstrings of length n that do not contain three consecutive 1's, but end with 01. Finally, let V_n represent bitstrings of length n that do not contain three consecutive 1's, but end with a 0.

   Observations:

   - Adding a 1 or a 0 to something counted in S_n yields a bitstring of length $n+1$ with three or more consecutive 1's. Adding a 1 to a string counted in T_n will also produce a bitstring of length $n+1$ with three consecutive 1's. If a bitstring ends in 01 or 0 there is no way to extend it to have three consecutive 1's. Hence,
$$S_{n+1} = 2S_n + T_n.$$
   - Starting with a string counted in U_n we get something in T_{n+1} by adding a 1 on the end, so
$$T_{n+1} = U_n.$$
   - Starting with a string counted in V_n we get something in U_{n+1} by adding a 1 on the end, so
$$U_{n+1} = V_n.$$
   - If we add a zero to something in T_n, U_n, or V_n we get something in V_{n+1}, so
$$V_{n+1} = T_n + U_n + V_n.$$

   The next step is to notice that $T_n = U_{n-1} = V_{n-2}$, so that

$$S_{n+1} = 2S_n + V_{n-2}.$$

   Similarly, the last equation reduces to $V_{n+1} = V_{n-2} + V_{n-1} + V_n$. It remains to appropriately initialize these two recursions ($n = 1$, $n = 2$, $n = 3$) and carry the calculations out to $n = 10$.

n	1	2	3	4	5	6	7	8	9	10
S_n	0	0	1	3	8	20	47	107	238	520
V_n	1	0	4	7	13	24	44	81	149	

   This shows that 520 bitsrings of length 10 have at least three consecutive 1's. □

4. A vector $x = (x_1, x_2, \dots, x_n)$ is said to be *s-sparse* if at most s of the coordinates x_j $(1 \le j \le n)$ are nonzero. Solve the following system of linear equations given the fact that $x = (x_1, \dots, x_5)$ is 2-sparse.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ -1 & 1 & 1 & -2 & -1 \\ -1 & -2 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \\ -3 \end{bmatrix}$$

Solution: We can row reduce the matrix to, hopefully, clean the problem up a bit.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 & -1 \\ -1 & 1 & 1 & -2 & -1 & 5 \\ -1 & -2 & 1 & 0 & -2 & -2 \\ 0 & -1 & 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 4 & -1 \\ 0 & 2 & 2 & -1 & 3 & 4 \\ 0 & -1 & 2 & 1 & 2 & -3 \\ 0 & -1 & 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 & -1 & -2 \\ 0 & 1 & 4 & 0 & 5 & 1 \\ 0 & 0 & 6 & 1 & 7 & -2 \\ 0 & 0 & 5 & 1 & 6 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

A 1-sparse solution would have to involve x_5 (since it is the only variable to appear in multiple equations), but the second equation would require $x_5 = 1$ and the fourth would require $x_5 = -2$. Hence, there is no 1-sparse solution.

A 2-sparse solution can only involve x_2, x_4, and x_5 since those are the only coordinates with nonzero coefficients in the two rows with nonzero entries in the last column. If x_5 is nonzero, then the first and third equations cannot be satisfied, so only x_2 and x_4 can be nonzero. The second equation forces $x_2 = 1$, while the fourth forces $x_4 = -2$. The only 2-sparse solution is thus

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

□

5. *Texas Hold'em* is a form of Poker in which each player is dealt two cards and all players share community cards that are dealt to the table. After the players receive their two cards, the community cards are dealt in three steps. First, the dealer places three cards on the table, which are referred to as the *flop*. Second, the dealer places a fourth card on the table, known as the *turn*. Finally, the dealer places a fifth card on the table, which is called the *river*. (Texas Hold'em uses a standard 52-card deck.)

Suppose that a player receives the Four of Clubs and the Five of Diamonds at the start of the game. Using the community cards, this player hopes to make a *straight*, which is a hand consisting of five consecutive cards ranging from `A-2-3-4-5` up to `10-J-Q-K-A`.

a. What is the probability that the player can form a straight after the flop has been dealt?

Solution: There are four different straights that we could hope to complete: `A-2-3-4-5`, `2-3-4-5-6`, `3-4-5-6-7`, and `4-5-6-7-8`. In each case, we need three specific cards and each of those appears four times in the 50 cards left in the deck. The number of ways a straight can be formed is thus 4^4 out of $\binom{50}{3}$ ways to deal three cards from the remaining 50, so the probability of a straight after the flop is

$$\frac{4^4}{\binom{50}{3}} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{50 \cdot 49 \cdot 48} = \frac{16}{49 \cdot 25} \approx 0.0130612.$$

This probability is often reported as a 1-in-76 chance in less formal discussions of Texas Hold'em, since $49 \times 25 \div 16 = 76.5625$. □

b. What is the probability that the player can form a straight (using the original two cards) after all five community cards have been dealt?

Solution: The five community cards may allow the player multiple options to form a straight and care must be taken not to double count these situations. The total number of ways to deal the five cards is $\binom{50}{5} = 2,118,760$ and we will systematically enumerate all the ways the various straights can occur.

A. `A-2-3-4-5-?-?`: This straight requires `A-2-3` to be dealt. There are four of each these cards, so there are 38 cards remaining in the deck that are not an Ace, Two, or Three. This gives $4^3\binom{38}{2}$ ways to complete the set of seven without using more than one Ace, Two, or Three. Another possibility is that we have one duplicate Ace/Two/Three and one card from the remaining 38, which leads to $3 \cdot 6 \cdot 4^2 \cdot 38$ more possibilities. If we have two duplicate Ace/Two/Three's (for a total of three), then there are $3 \cdot 4^3$ more possibilities. Finally, we could have duplicates of two different kinds of cards, leading to $6 \cdot 6 \cdot 4 \cdot 3$ more possibilities. This category of situations leads to a total of

$$N_1 = 4^3 \cdot \binom{38}{2} + 3 \cdot 6 \cdot 4^2 \cdot 38 + 3 \cdot 4^3 + 6 \cdot 6 \cdot 4 \cdot 3$$

ways to form a straight `A-2-3-4-5-?-?`.

B. `2-3-4-5-6-?-?` (no Aces): The analysis of this case is similar to the previous one except for the exclusion of Aces. That means the deck essentially has 34 cards and this situation leads to

$$N_2 = 4^3 \cdot \binom{34}{2} + 3 \cdot 6 \cdot 4^2 \cdot 34 + 3 \cdot 4^3 + 6 \cdot 6 \cdot 4 \cdot 3.$$

Note that each instance counted by N_1 has an Ace, so there is no overlap between these categories.

C. `3-4-5-6-7-?-?` (no Twos): This case is almost identical to the previous one. The only difference is that we must remove the Twos from the deck and put the Aces back, so $N_3 = N_2$.

D. `4-5-6-7-8-?-?` (no Threes): This time we remove the Threes from the deck and replace the Twos, so that $N_4 = N_2$.

The probability of a straight (using the Four-Five) after all five community cards have been dealt is thus

$$\frac{N_1 + N_2 + N_3 + N_4}{2,118,760} = \frac{56,560 + 3 \cdot 46,320}{2,118,760} = \frac{195,520}{2,118,760} = \frac{104}{1,127} \approx 0.092280.$$

We have excluded the possibility that a straight can be formed entirely from the community cards, since all players have the same hand in that situation. □