SLU Missouri Collegiate Math Team – Mock Exam 2017
March 25, 2017
Explain your work carefully to maximize credit.

Problems:

1. The number 99 was multiplied by an integer k to obtain an integer of seven decimal digits, but two of the digits got blotted out on the paper. The product was $62ab427$, but the digits a and b are illegible. Determine all possible values of a and b.

   Source: Iowa Collegiate Mathematics Competition, 2017.

2. Find the sum of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots,$$

   where the terms are the reciprocals of the positive integers whose only prime factors are twos and threes.

   Source: Missouri Collegiate Mathematics Competition, 2014.

3. Notice that 73 can be written as a sum of two consecutive positive integers, $73 = 36 + 37$. Prove that no longer sum of consecutive positive integers equals 73.

4. Suppose that the function f satisfies $f'(x) = 1 + f(x)$ for all x. If $f(2) = 3$ find:

   (a) $f^{(10)}(2)$ where $f^{(10)}$ denotes the 10th derivative of f;

   (b) $f(3)$.

   Source: Iowa Collegiate Mathematics Competition, 2017.

5. For real $a > 0$ define the sequence $\{x_n\}$ by

$$x_{n+1} = a(x_n^2 + 4), \quad x_0 = 0.$$

   Determine necessary and sufficient conditions on a for $\lim_{n\to\infty} x_n$ to exist and be finite.

   Source: Missouri Collegiate Mathematics Competition, 2012.

1. Since the product ends in 27, the product is 73 less than a multiple of 100. This means $k \equiv 73 \mod 100$.

Also

$$\begin{array}{r} 62626 \\ 99\,\overline{)\,6200000} \\ 594 \\ 26000 \\ 198 \\ 620 \end{array}$$

so $k \geq 62626$ and

$$\begin{array}{r} 63514 \\ 99\,\overline{)\,6300000} \\ 594 \\ 360 \\ 299 \\ 610 \\ 595 \\ 150 \\ 99 \\ 51 \end{array}$$

Hence, $62626 \leq k \leq 63514$.

The first possibility is 62673:

$$\begin{array}{r} 62673 \\ \times\ 99 \\ \hline 564057 \\ 564057 \\ \hline 6204627 \end{array}$$

Other possibilities add 100, 200, etc. to this which increases the product by 9900 each time:

6204627	6254127
6214527	6264027
$\boxed{6224427}$	6273927
6234327	6283827
6244227	6293727

Only 6224427 fits, so ab can only be 24. $\square$

2. Let $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$

Then

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \frac{1}{24} + \cdots$$

which includes any terms in S except reciprocals of 3^k, $k \geq 0$.

Hence,

$$\frac{S}{2} = S - \left[1 + \frac{1}{3} + \frac{1}{9} + \cdots\right] = S - \frac{1}{1 - 1/3} = \frac{3}{2}$$

which implies

$$\frac{S}{2} = \frac{3}{2} \quad \text{and} \quad S = 3.$$

$\square$

$\boxed{3.}$ If 73 is a sum of k consecutive positive integers starting with x then

$$x+(x+1)+\cdots+(x+k-1) = kx + \frac{(k-1)k}{2} = 73.$$

If this equation holds then $73 - \frac{(k-1)k}{2} \equiv 0 \mod k$, which we can check without knowing x.

Notice if k is odd then $kx + \frac{(k-1)k}{2} \equiv 0 \mod k$, while $73 \not\equiv 0 \mod k$ for $k \geq 3$ because 73 is prime. So odd $k \geq 3$ is impossible.

Notice if $x=1$ and $k=12$ then $kx + \frac{k-1}{2}k = 78 > 73$. So only $k \leq 10$ need be considered.

Finally, we compute $73 - \frac{(k-1)k}{2} \mod k$ for $k = 4, 6, 8, 10$.

k	4	6	8	10
$\mathrm{mod}(73 - \frac{(k-1)k}{2}, k)$	3	4	5	8

Since none are zero, 73 cannot be written as a sum of k consecutive positive integers for any $k > 2$.

$\boxed{4.}$ Let $y = f(x)$ and rewrite the DE as $\frac{dy}{dx} - y = 1$.
The characteristic equation is $r - 1 = 0$ so e^x is a homogeneous solution.
Using the method of undetermined coefficients we assume $y(x) = b$ is a particular solution:

$$(b)' - b = 1 \longrightarrow b = -1.$$

Thus the general solution is $y(x) = Ae^x - 1$.
Since $y(2) = 3$, we find that $3 = Ae^2 - 1$ so $Ae^2 = 4$. $(A = 4e^{-2})$

(a) Since $f(x) = 4e^{-2}e^x - 1 \Longrightarrow f^{(10)}(x) = 4e^{-2}e^x \Longrightarrow f^{(10)}(2) = 4$.
(b) $f(3) = 4e^{-2}e^3 - 1 = 4e - 1$.

$\square$

$\boxed{5.}$ If a limit exists then $x = ax^2 + 4a$ so $ax^2 - x + 4a = 0$ has a real solution.
Thus a necessary condition is $1 - 16a^2 > 0$ or $a^2 < \frac{1}{16}$ or $0 < a < \frac{1}{4}$.

If $0 < a < \frac{1}{4}$ then the solutions are $x = \frac{1}{2a} \pm \frac{\sqrt{1-16a^2}}{2a}$ so $0 < x < \frac{1}{a}$.

① $x_{n+1} - x_n = a(x_n^2 - x_{n-1}^2)$ $\quad x_1 - x_0 = 4a > 0$ so induction implies $x_{n+1} \geq x_n$.
② $x_1 = 4a < 2$

$$x_{n+1} = a(x_n^2 + 4) \leq \tfrac{1}{4}(4+4) = 2 \quad \text{so } x_n \leq 2.$$

Since x_n is bounded, monotonic it must converge.

$\square$