SLU Missouri Collegiate Math Team – Qualifying Exam 2016

April 3, 2016

Explain your work carefully.

Problems:

1. (Iowa Collegiate Math Competition, 2015) We are given six jugs, the first five containing 2 liters of water each, and the sixth containing one liter. At each step we can select any two jugs, and then pour water from one into another until they cntain equal amounts of water. Is it possible to make the quantitites of water in all jugs equal? Explain you answer.

2. A bag contains 10 balls, numbered 0-9. Three balls are drawn: b_1, b_2, and b_3. What is the probability that $b_1 < b_2 < b_3$?

3. Compute $\displaystyle\int_0^\infty \frac{\ln x}{x^2 + 2x + 4}\, dx$.

4. (Iowa Collegiate Math Competition, 2015) Find all right triangles whose sides are positive integers and whose perimeter is numerically equal to its area.

5. Let N be the random variable representing the number of coin flips required for two consecutive flips to have the same result. Assuming the coin is fair, what is the expected value of N?

1. Let $m_{j,1} \leq m_{j,2} \leq \cdots \leq m_{j,6}$ be the amounts in the jugs after j steps, in increasing order.

Equalize 1 with 4, 2 with 5, and 3 with 6 over 3 steps.

Then $m_{j+3,1} = \dfrac{m_{j,1} + m_{j,4}}{2}$ and $m_{j+3,6} = \dfrac{m_{j,3} + m_{j,6}}{2}$

so $m_{j+3,6} - m_{j+3,1} = \underbrace{\dfrac{m_{j,6} - m_{j,1}}{2}}_{} + \underbrace{\dfrac{m_{j,3} - m_{j,4}}{2}}_{\leq 0} \leq \dfrac{1}{2}\left(m_{j,6} - m_{j,1}\right)$

so the jugs will converge geometrically to a state in which all of the jugs contain $\frac{11}{6}$ gallons. □

2. The number of orderings is $\dfrac{10!}{(10-3)!} = 720$.

The number which occur in increasing order is

$$\sum_{b_1=0}^{7} \sum_{b_2=b_1+1}^{8} \sum_{b_3=b_2+1}^{9} 1 = \sum_{b_1=0}^{7} \sum_{b_2=b_1+1}^{8} 9 - b_2$$

$$= \sum_{b_1=0}^{7} \left[9(8-b_1) + \dfrac{b_1(b_1+1)}{2} - \dfrac{8(9)}{2} \right]$$

$$= \sum_{b_1=0}^{7} \left(36 - 8.5 b_1 + 0.5 b_1^2 \right)$$

$$= 288 - 8.5\left(\dfrac{7(8)}{2}\right) + 0.5 \cdot \dfrac{7(8)(15)}{6}$$

$$= 288 - 8.5(28) + 14(5)$$

$$= 120$$

Probability $= \dfrac{120}{720} = \dfrac{1}{6}$ □

3. $\displaystyle\int_0^\infty \dfrac{\ln x}{x^2+2x+4}\,dx \underset{\substack{u=\frac{4}{x} \\ dx = -\frac{4}{u^2}du}}{=} \int_\infty^0 \dfrac{\ln 4/u}{\frac{16}{u^2} + \frac{8}{u} + 4} \cdot \dfrac{-4}{u^2}\,du = \int_0^\infty \dfrac{\ln 4 - \ln u}{u^2 + 2u + 4}\,du$

Hence, $\displaystyle 2\int_0^\infty \dfrac{\ln x}{x^2+2x+4}\,dx = \int_0^\infty \dfrac{1}{3}\dfrac{\ln 4}{1+\left(\frac{x+1}{\sqrt3}\right)^2}\,dx = \dfrac{1}{\sqrt3}\arctan\left(\dfrac{x+1}{\sqrt3}\right)\Big|_0^\infty = \dfrac{2\ln 2}{\sqrt3}\left(\dfrac{\pi}{2} - \dfrac{\pi}{6}\right)$

so it follows that

$$\int_0^\infty \dfrac{\ln x}{x^2+2x+4}\,dx = \dfrac{\ln 2}{3\sqrt3}\pi$$ □

4.
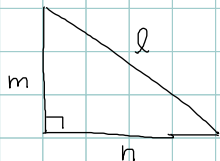
Perimeter $= m + n + \ell$ $\ell = \sqrt{m^2 + n^2}$

Area $= \frac{1}{2}mn$

$\frac{1}{2}mn - m - n = \sqrt{m^2 + n^2} \longrightarrow \frac{1}{4}m^2n^2 - m^2n - mn^2 + \cancel{m^2} + 2mn + \cancel{n^2} = \cancel{m^2} + \cancel{n^2}$

$\longrightarrow \frac{1}{4}mn - m - n + 2 = 0$

$\longrightarrow m(n-4) = 4n - 8 \longrightarrow m = \dfrac{4(n-2)}{(n-4)}$

In order for m to be an integer $n-4$ must divide $4(n-2)$.
This yields $n = 5, 6, 8, 12$:

$n = 12$	$m = 5$	$\ell = 13$	Area $= 30$	Perimeter $= 30$
$n = 5$	$m = 12$	$\ell = 13$	$= 30$	30
$n = 6$	$m = 8$	$\ell = 10$	$= 24$	24
$n = 8$	$m = 6$	$\ell = 10$	$= 24$	24

Since $\dfrac{4(n-2)}{n-4} \to 4$ there won't be any other solutions because $n > 12 \Rightarrow m < 5$. ☐

5.

$P(N=2) = \frac{1}{2}$ 00 or 11

$P(N=3) = P(N>2) \cdot P(\text{flip } 3 = \text{flip } 2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

<u>Claim</u>: $P(N=k) = 2^{-(k-1)}$

<u>PF</u>: If $P(N=k) = 2^{-(k-1)}$ for $2 \le k \le n$ then

$$P(N=k+1) = P(N>k) \cdot P(\text{flip } k+1 = \text{flip } k)$$
$$= \left(1 - \left(\frac{1}{2} + \frac{1}{4} + \cdots + 2^{-(n-1)}\right)\right) \cdot \frac{1}{2}$$
$$= 2^{-(n-1)} \cdot \frac{1}{2}$$
$$= 2^{-(n+1-1)} \qquad \text{☐}$$

Now,

$$E[N] = \sum_{k=2}^{\infty} k\, P(N=k) = \sum_{k=2}^{\infty} k\, 2^{1-k}$$

Recall that $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ so $\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1}$

Hence,

$$E[N] = \sum_{k=2}^{\infty} k\, 2^{1-k} = \frac{1}{(1-\frac{1}{2})^2} - 1 = 3 \qquad \text{☐}$$