SLU Math Team 2014 Qualifying Problems

Return your work to Dr. Lamar's office (Ritter 213) before 4pm on Wednesday, March 19. Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) to declare your desire to participate in the contest.

1. For $\theta = 10^{2014}$ degrees, compute $\cos(\theta^\circ)$ to two decimal places.

   Solution. $\cos(\theta^\circ) = 0.17$ to two decimal places.

   First, $10^n \equiv 280 \pmod{360}$ for any $n \geq 3$, so

   $$\cos(\theta^\circ) = \cos(280^\circ) = \sin(10^\circ) = \sin(\pi/18).$$

   Using the MacLaurin series for $\sin(x)$, $x - x^3/6 < \sin(x) < x$. An easy hand computation shows $\pi/18 \approx 0.174$ and $0.174^3/6 < 0.002$. Then $0.172 < \sin(\pi/18) < 0.175$. $\square$

2. Find the units digit of $\displaystyle\sum_{n=0}^{99} n!$.

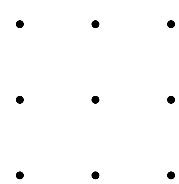
   Solution. The units digit is 4.

   Since $n!$ is divisible by 10 for $n \geq 5$, the sum in question has the same units digit as

   $$\sum_{n=0}^{4} n! = 0! + 1! + 2! + 3! + 4! = 1 + 1 + 2 + 6 + 24 = 34$$

   $\square$

3. Nine dots are arranged on a square grid as shown below. Find three triangles so that each dot lies on exactly one triangle, and each triangle touches exactly three dots. Or, prove it cannot be done.

   Solution. It can be done: for example, use triangles (0,0)-(1,0)-(0,1) and (1,1)-(2,2)-(1,2) and (0,2)-(2,1)-(2,-1). Note that this cannot be done if the triangle corners are restricted to the dots. There are four corner dots to the square. One of the three triangles must contain two of these dots. If the dots are required to be corners of the triangles, then the two corner dots have a segment joining them and that segment also hits another dot, violating each triangle touches three dots. $\square$

4. Find all real values of $a > 0$ such that $a^x \geq x^a$ holds for all $x > 0$.

   Solution. The only solution is $a = e$.

   Clearly, $a^x = x^a$ when $x = a$. At $x = a$,

$$\frac{d}{dx} a^x \bigg|_{x=a} = (\log a) a^a \quad \text{and} \quad \frac{d}{dx} x^a \bigg|_{x=a} = a^a$$

   If these two derivatives are unequal, then the graphs of x^a and a^x cross at $x = a$ and so $a^x \geq x^a$ cannot hold for x near a. Therefore, it is necessary that $\log a = 1$, or $a = e$.

   Now, claim that $e^x \geq x^e$ for all $x > 0$.

$$e^x \geq x^e \iff x \geq e \log(x) \iff \frac{1}{e} \geq \frac{\log(x)}{x}.$$

   Now let $f(x) = \frac{\log(x)}{x}$. Then

$$f'(x) = \frac{(1 - \log(x))}{x^2}$$

   and $f'(x) = 0$ if and only if $x = e$. Now

$$f''(e) = \frac{-e - 2e(1 - \log(e))}{e^4} = -\frac{1}{e^3} < 0.$$

   So f takes its maximum value on $(0, \infty)$ at $x = e$, so that $f(e) \geq f(x)$ for all $x > 0$. □

5. Compute

$$\int_0^\infty \frac{\log(x)}{1 + x^2} dx.$$

   Solution. Let I be the integral in question. Change variables $x = 1/y$, so $dx = -dy/y^2$. Then

$$I = \int_\infty^0 \frac{-\log(y)}{1 + (1/y)^2} \frac{-dy}{y^2} - \int_0^\infty \frac{\log(y)}{1 + y^2} dy = -I.$$

   Since $I = -I$, the integral is zero.

   To fully justify the above computation, one needs to know the function $\log(x)/(1 + x^2)$ is absolutely integrable on $(0, \infty)$. For $x \geq 1$,

$$\left| \frac{\log(x)}{1 + x^2} \right| < \frac{\sqrt{x}}{x^2} = x^{-3/2},$$

   which is integrable on $(1, \infty)$. For $x < 1$,

$$\left| \frac{\log(x)}{1 + x^2} \right| < \left| \log(x) \right|,$$

   which is integrable on $(0, 1)$.

   □