SLU Math Team 2013 Qualifying Problems

Return your work to Dr. Clair's office (Ritter 110) before 4pm on Tuesday, April 2. Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) to declare your desire to go on the trip.

1. In the product $1! \cdot 2! \cdot 3! \cdot \dots \cdot 99! \cdot 100!$, prove that erasing one of the factors of the product results in a perfect square.

   Solution. In the product, 1 appears 100 times, 2 appears 99 times, 3 appears 98 times, and so on. Then

   $$\begin{aligned} 1!{\cdot}2! \cdot 3! \cdot \dots \cdot 99! \cdot 100! &= \\ &= 1^{100} \cdot 2^{99} \cdot 3^{98} \cdot \dots \cdot 100^1 \\ &= \left(1^{50} \cdot 2^{49} \cdot 3^{49} \cdot 4^{48} \cdots 98^1 \cdot 99^1\right)^2 \cdot 2 \cdot 4 \cdot 6 \cdot \dots \cdot 98 \cdot 100 \\ &= \left(1^{50} \cdot 2^{49} \cdot 3^{49} \cdot 4^{48} \cdots 98^1 \cdot 99^1\right)^2 \cdot 2^{50} \cdot 50! \end{aligned}$$

   So, erasing the term 50! from the product results in a square. □

2. Two positive numbers real numbers are given. Their sum is less than their product. Prove their sum is at least 4.

   Solution. Call the numbers x and y, so $x + y < xy$ is given. Then

   $$(x+y)^2 = x^2+2xy+y^2 = x^2-2xy+y^2+4xy = (x-y)^2+4xy \geq 4xy > 4(x+y).$$

   Dividing by $(x + y)$ gives $(x + y) > 4$.

   Note that the middle few steps are essentially the proof of the AMGM inequality, and one could simply apply AMGM instead. □

3. Let $f : [1, 2] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_1^2 f(x)dx = 0$.

   Prove there is a number $c \in (1, 2)$ such that $\displaystyle cf(c) = \int_c^2 f(x)dx$.

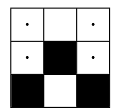
   Solution. Let $F(t) = t \int_t^2 f(x)dx$, for $t \in [1, 2]$. F is continuous on $[1, 2]$, differentiable on $(1, 2)$, and $F(1) = F(2) = 0$. By Rolle's Theorem, there is $c \in (1, 2)$ with $F'(c) = 0$.

   Compute $F'(t) = \int_t^2 f(x)dx - tf(t)$. When $t = c$, this gives

   $$0 = \int_c^2 f(x)dx - cf(c).$$

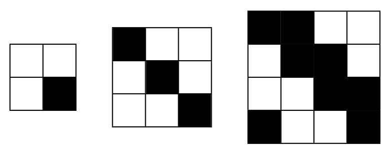
   □

4. Imagine a board made out of squares, with each square colored either black or white. Say that a board is *wrecked* if no rectangular sub-board has all four of its corners the same color. For example, the board below is not wrecked because the dots form a rectangle with four white corners:

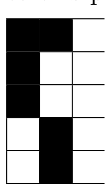


For which integers $n > 1$ is there an $n \times n$ square board which is wrecked?

Solution. Only for $n = 2, 3, 4$. It's not hard to see that $n = 2, 3, 4$ can be wrecked:



If an $n \times n$ square can be wrecked, then so can any smaller square (since the smaller square appears inside the larger). Assume now that $n = 5$ and that the 5×5 square is wrecked. Say a column is light if it has 3 or more white squares, and dark if it has 3 or more black squares. Every column is either dark or light, and there are 5 columns, so by the pigeonhole principle, at least three columns are dark or at least three colums are light. Without loss of generality (by switching black with white), assume there are at least three dark columns. If a square is wrecked, then it is still wrecked if one interchanges any pair of rows or columns. By switching columns, assume the three dark columns are the first three columns. Since columns 1 and 2 have three black squares, there must be a row in which columns 1 and 2 are both black. Switching rows around, we may assume that columns 1 and 2 have black squares in the first row. Also, rearrange so that column 1 is black in row 2 and row 3. Since the square is wrecked, column 2 must be white in rows 2 and 3, and black in rows 4 and 5, leading to this picture:



Finally, column 3 needs at least three black squares. If one is in row 1, then any of the other black squares in column three finishes a rectangle with all black corners. If column 3, row 1 is white, then column 3 either has black squares in both rows 2 and 3 or black squares in both rows 4 and 5, finishing a rectangle with column 1 in the first case and column 2 in the second.

This contradicts the assumption that the original 5×5 square was wrecked.

So, for $n \geq 5$, an $n \times n$ square can never be wrecked.

$\square$

5. Is the following series convergent or divergent?

$$\frac{1}{2} \cdot \frac{19}{7} + \frac{2!}{3^2} \cdot \left(\frac{19}{7}\right)^2 + \frac{3!}{4^3} \cdot \left(\frac{19}{7}\right)^3 + \frac{4!}{5^4} \cdot \left(\frac{19}{7}\right)^4 + \cdots$$

Solution. Apply the ratio test, dividing the n^{th} term by the $n-1^{st}$ term:

$$\frac{n!}{(n+1)^n} \left(\frac{19}{7}\right)^n \times \left(\frac{(n-1)!}{n^{n-1}} \left(\frac{19}{7}\right)^{n-1}\right)^{-1} = n \cdot \frac{19}{7} \cdot \frac{n^{n-1}}{(n+1)^n}$$
$$= \frac{19}{7} \cdot \left(\frac{n}{n+1}\right)^n = \frac{19}{7} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{19}{7e}$$

as $n \to \infty$. Now $7e \approx 19.028$, so $\frac{19}{7e} < 1$ and the series converges (to approximately 1572.88). $\square$