SLU Math Team 2012 Qualifying Problems

Return your work to Dr. Clair's office (Ritter 110) before 4pm on Monday, April 2. Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) to declare your desire to go on the trip.

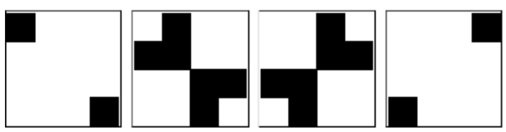
1. Find all positive integers k so that

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \cdots + \frac{9}{k}$$

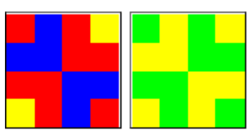
is an integer.

Solution. The sum is $(1 + 2 + \cdots + 9)/k = 45/k$. This is an integer when $k = 1, 3, 5, 9, 15, 45$. □

2. There is a two-player game called *Continuo* which is played with a deck of colored square cards. Each card has four zones, as shown in the picture below, and notice that each zone consists of two separate regions.



Each zone can be colored red, blue, yellow, or green, but no two zones that share an edge are allowed to be the same color, and no card can use all four colors. Here are two example cards:



How many different *Continuo* cards are there?

Solution. There are four possible colors for zone 1, which leaves three choices for zone 2, then three choices for zone 3, then three choices for zone 4. So there are 108 ways to color the zones so no adjacent zones have the same color. However, one is not allowed to use all four colors on a card. There are $4 * 3 * 2 * 1 = 24$ ways to use all four colors, which gives $108 - 24 = 84$ cards that meet the coloring criteria. However, there is a $90°$ rotational symmetry which interchanges zone 1 with zone 4, and zone 2 with zone 3. So only half of these colorings give different cards. There are 42 different *Continuo* cards. □

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, with $f(f(x)) \cdot f(x) = 1$ for all $x \in \mathbb{R}$. Suppose $f(1024) = 512$. Find $f(256)$.

Solution. If y is in the range of f, say $y = f(x)$, then $f(y) \cdot y = 1$ so $f(y) = 1/y$. Now 512 is in the range of f, and so $f(512) = 1/512$ so $1/512$ is also in the range of f. Since f is continunous, every value in $[1/512, 512]$ is also in the range of f (by the Intermediate Value Theorem). Therefore 256 is in the range of f and so $f(256) = 1/256$. $\square$

4. Compute $\displaystyle\lim_{n\to\infty} \prod_{k=0}^{n-1} \cos\left(\frac{2^k}{n!}\right)$

Solution. First, establish the identity:

$$\cos(x)\cos(2x)\cos(4x)\cdots\cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin(x)} \tag{1}$$

by induction. When $n = 1$, the double angle identity gives

$$\frac{\sin(2x)}{2\sin(x)} = \cos(x)$$

and then for the inductive step,

$$\frac{\sin(2^n x)}{2^n \sin(x)} = \frac{2\sin(2^{n-1}x)\cos(2^{n-1}x)}{2^n \sin(x)} = \frac{\sin(2^{n-1}x)}{2^{n-1}\sin(x)}\cos(2^{n-1}x).$$

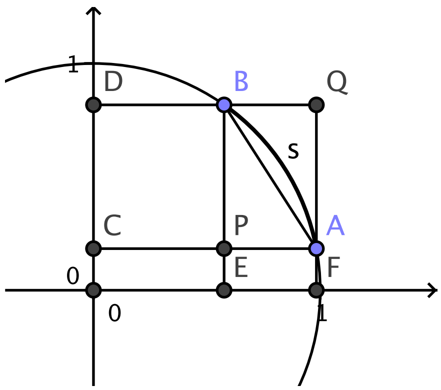
Then using (1) with $x = 1/n!$,

$$\begin{aligned} \lim_{n\to\infty} \prod_{k=0}^{n-1} \cos\left(\frac{2^k}{n!}\right) &= \lim_{n\to\infty} \frac{\sin(2^n/n!)}{2^n \sin(1/n!)} \\ &= \lim_{n\to\infty} \frac{\sin(2^n/n!)}{2^n/n!} \frac{1/n!}{\sin(1/n!)} \\ &= 1 \end{aligned}$$

since $\lim_{n\to\infty} 2^n/n! = 0$ and $\lim_{\theta\to 0} \sin(\theta)/\theta = 1$. $\square$

5. Let s be any arc of the unit circle lying entirely in the first quadrant. Let G be the area of the region lying below s and above the x-axis. Let H the area of the region lying to the right of the y-axis and to the left of s. Prove that the sum of these areas, $G + H$, depends only on the arc length, and not on the position, of s.

Solution. Suppose the arc s runs from point A to point B. The area K between the chord AB and the unit circle depends only on the length of s.



The total area is $G+H = 2K + \text{area}(ABEF) + \text{area}(ABDC)$. The triangle ABP is covered twice, and has the same area as triangle ABQ, so that $\text{area}(ABEF) + \text{area}(ABDC) = \text{area}(0FQD) - \text{area}(0EPC)$.

Now let $A = (\cos(\alpha), \sin(\alpha))$ and $B = (\cos(\beta), \sin(\beta))$, where $0 \leq \alpha < \beta \leq \pi/2$. Then

$$G + H = 2K + \cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) = 2K + \sin(\beta - \alpha)$$

which depends only on the length of s. $\square$