SLU Math Team 2012 Qualifying Problems

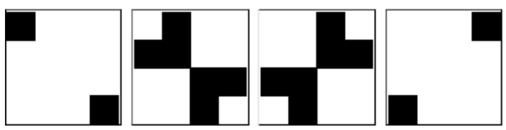
Return your work to Dr. Clair's office (Ritter 110) before 4pm on Monday, April 2. Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) to declare your desire to go on the trip.

1. Find all positive integers k so that

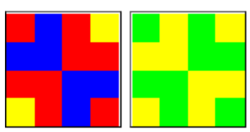
$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \cdots + \frac{9}{k}$$

is an integer.

2. There is a two-player game called *Continuo* which is played with a deck of colored square cards. Each card has four zones, as shown in the picture below, and notice that each zone consists of two separate regions.



Each zone can be colored red, blue, yellow, or green, but no two zones that share an edge are allowed to be the same color, and no card can use all four colors. Here are two example cards:



How many different *Continuo* cards are there?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, with $f(f(x)) \cdot f(x) = 1$ for all $x \in \mathbb{R}$. Suppose $f(1024) = 512$. Find $f(256)$.

4. Compute $\displaystyle\lim_{n\to\infty} \prod_{k=0}^{n-1} \cos\left(\frac{2^k}{n!}\right)$

5. Let s be any arc of the unit circle lying entirely in the first quadrant. Let G be the area of the region lying below s and above the x-axis. Let H the area of the region lying to the right of the y-axis and to the left of s. Prove that the sum of these areas, $G + H$, depends only on the arc length, and not on the position, of s.