SLU Math Team 2011 Qualifying Problems

Return your work to Dr. Clair on or before Tuesday, March 22. Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) to declare your desire to go on the trip.

1. Let N be the product of all positive integers which divide 1,000,000,000. Find $\log_{10}(N)$.

    Solution. The solution is 450. $10^9 = 2^9 5^9$, and divisors are of the form $2^a 5^b$ with $0 \le a \le 9$ and $0 \le b \le 9$. Then there are 100 divisors of 10^9, and they occur in pairs $d, \frac{10^9}{d}$ which multiply to 10^9. So $N = (10^9)^{50} = 10^{450}$ and $\log_{10}(N) = 450$. In general, for 10^k the value is $\frac{1}{2}k(k+1)^2$. $\square$

2. Evaluate $\displaystyle\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{\pi/2}} \sin(x^2)\,dx\,dy$.

   *Solution.* The solution is 1. Interchange the order of integration to get

    $$\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{\pi/2}} \sin(x^2)\,dx\,dy = \int_0^{\sqrt{\pi/2}} \int_0^{2x} \sin(x^2)\,dy\,dx$$

    $$= \int_0^{\sqrt{\pi/2}} 2x \sin(x^2)\,dx$$

    $$= -\cos(x^2)\Big|_0^{\sqrt{\pi/2}} = 1.$$

    $\square$

3. Find a real valued function f so that $f(f(x)) = -\frac{1}{x}$ for all positive x.

    Solution. Let $f(x) = \frac{x-1}{x+1}$, and it is easy to check that the equation is satisfied. To find this function, you might guess that $f(x) = \frac{ax+b}{cx+d}$ and then compute $f(f(x))$ and solve for a, b, c, d. In fact, there is a theory of such functions, called Mobius transformations, which makes it clear how to solve many problems of this form. $\square$

4. Consider the hexagon shown below with 120° angles, short sides of length 3, and long sides of length 5. Can it be tiled (covered without overlaps) by rhombuses with side length 1 and angles $60° - 120° - 60° - 120°$?
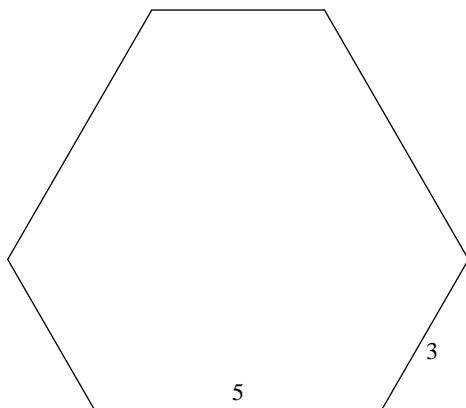
Solution. It cannot be done. Divide the hexagon into 94 equilateral tri-
angles, each of side length 1. Color these triangles black and white in
a checkerboard pattern, so no triangles of the same color share an edge.
There are 48 of one color and 46 of the other. Since each rhombus covers
one triangle of each color, there is no way to tile the hexagon. $\qquad\square$

5. Take two rays from O making an angle $0 < \theta < \pi$. Let C_1 be the circle
   which is tangent to both rays and has center at distance 1 from O. Let C_2
   be the circle which is tangent to both rays and C_1, and which is smaller
   than C_1. Find θ so that C_2 is a large as possible.

   Solution. Let $\alpha = \theta/2$, and let r be the radius of C_2. Let P_1 and P_2 be
   the centers of C_1 and C_2, respectively, and let Q_1 and Q_2 be the points of
   tangency of circles C_1 and C_2 with one of the rays.

   Then OP_1 has length 1, and P_1Q_1 has length $\sin(\alpha)$. Also, OP_2 has length
   $1 - \sin(\alpha) - r$ and P_2Q_2 has length r. Since triangles OP_1Q_1 and OP_2Q_2
   are similar, we get
   $$\frac{1 - \sin(\alpha) - r}{1} = \frac{r}{\sin(\alpha)}.$$
   Solving for r gives
   $$r = \sin(\alpha)\frac{1 - \sin(\alpha)}{1 + \sin(\alpha)}$$
   Notice $r \geq 0$ with $r = 0$ only at the endpoint values of $\alpha = 0, \pi/2$.
   Compute
   $$\frac{dr}{d\alpha} = \frac{\cos(\alpha)}{(1 + \sin(\alpha))^2}\left(1 - 2\sin(\alpha) - \sin^2(\alpha)\right)$$
   The only solution to $dr/d\alpha = 0$ on the interval $(0, \pi/2)$ is when $1 - 2\sin(\alpha) - \sin^2(\alpha) = 0$, or $\sin(\alpha) = -1 + \sqrt{2}$. Thus
   $$\theta = 2\arcsin(-1 + \sqrt{2}) \approx 48.9°.$$
   $\qquad\square$