SLU Math Team 2010 Qualifying Problems

Return your work to Dr. Clair on or before Thursday, March 25.

Even if you feel you got none of the problems, you need to hand in something (a blank sheet of paper with your name on it?) if you want to go on the trip.

1. Calculate $\lim_{x \to 1} \frac{\int_x^{x^2} e^{-t^2}\, dt}{x-1}$.

   Solution. It is e^{-1}. Put
   $$F(x) = \int_x^{x^2} e^{-t^2}\, dt = \int_0^{x^2} e^{-t^2}\, dt - \int_0^x e^{-t^2}\, dt.$$
   By the Fundamental Theorem of Calculus and the chain rule,
   $$F'(x) = 2xe^{-(x^2)^2} - e^{-x^2}.$$
   Then, applying L'Hopital's rule,
   $$\lim_{x \to 1} \frac{\int_x^{x^2} e^{-t^2}\, dt}{x-1} = \lim_{x \to 1} \frac{F'(x)}{1} = F'(1) = e^{-1}.$$
   $\square$

2. Suppose that $r > 0$ is a rational approximation to $\sqrt{5}$. Prove that $\frac{2r+5}{r+2}$ is a better approximation to $\sqrt{5}$.

   Solution.
   $$\begin{aligned} \left| \frac{2r+5}{r+2} - \sqrt{5} \right| &= \left| \frac{2r+5-\sqrt{5}r-2\sqrt{5}}{r+2} \right| && (1) \\ &= \left| \frac{r(2-\sqrt{5}) - \sqrt{5}(2-\sqrt{5})}{r+2} \right| && (2) \\ &= \left| \frac{2-\sqrt{5}}{r+2} \right| \left| r - \sqrt{5} \right| && (3) \\ &< \left| r - \sqrt{5} \right|. && (4) \end{aligned}$$
   Where we used the fact that $\frac{\sqrt{5}-2}{r+2} < 1$ for $r > 0$ (in fact, it's true for $r > \sqrt{5} - 4$.) $\square$

3. Let P be a polygon whose vertices have integer coordinates and whose sides have integer lengths. Prove that the perimeter of P is an even number.

Solution. Let the vertices of the polygon be p_i, $i = 1, \ldots, n$. Let v_i be the vector from p_i to p_{i+1} (with v_n from p_n to p_1). Write $v_i = (a_i, b_i)$, and let c_i be the length of v_i. Since the polygon is closed, $\sum_{i=1}^n v_i = (0, 0)$, and so $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$.

Now, by the pythagorean theorem, $a_i^2 + b_i^2 = c_i^2$. Working modulo 2, $a_i^2 \equiv a_i \pmod 2$, $b_i^2 \equiv b_i \pmod 2$, and $c_i^2 \equiv c_i \pmod 2$, so that $a_i + b_i \equiv c_i \pmod 2$. Then the perimiter is

$$\sum_{i=1}^n c_i \equiv \sum_{i=1}^n a_i + b_i \pmod 2 \tag{5}$$

$$\equiv 0 \pmod 2. \tag{6}$$

So, the perimiter is even. □

4. Two distinct numbers a and b are chosen at random from the set $\{2, 2^2, 2^3, \ldots, 2^{25}\}$. What is the probability that $\log_b(a)$ is an integer?

Solution. Let $a = 2^j$ and $b = 2^k$. Then

$$\log_a b = log_{2j} 2k = \frac{\log 2^k}{\log 2^j} = \frac{k \log 2}{j \log 2} = \frac{k}{j},$$

so $\log_a b$ is an integer if and only if k is an integer multiple of j . For each j , the number of integer multiples of j that are at most 25 is $\left\lfloor \frac{25}{j} \right\rfloor$. Because $j \neq k$, the number of possible values of k for each j is $\left\lfloor \frac{25}{j} \right\rfloor - 1$. Hence the total number of ordered pairs (a, b) is

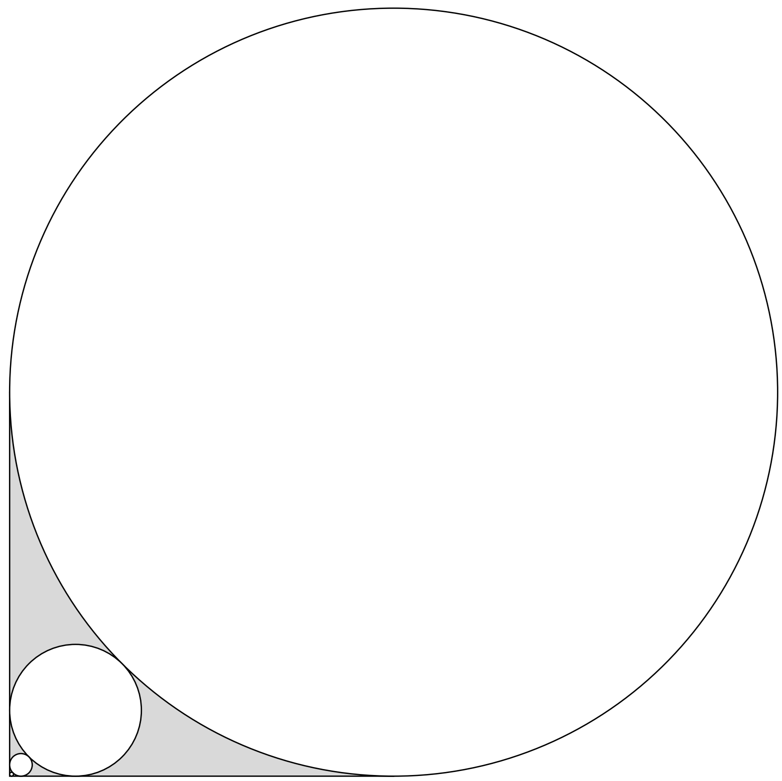
$$\sum_{j=1}^{25} \left(\left\lfloor \frac{25}{j} \right\rfloor - 1 \right) = 24 + 11 + 7 + 5 + 4 + 3 + 2(2) + 4(1) = 62.$$

Since the total number of possibilities for a and b is $25 \cdot 24$, the probability that $\log_a b$ is an integer is

$$\frac{62}{25 \cdot 24} = \frac{31}{300}.$$

□

5. In the picture below, the largest circle has radius 1, and there are infintely many smaller circles packed into the corner, each as large as possible. What is the area of the shaded region?



Solution. The area is $1 + \frac{\pi}{4} - \frac{3\pi\sqrt{2}}{8} \approx 0.119$.

Call the large circle C_0, the next smallest C_1, and so on. The center of the large circle is $(1, 1)$, and let C_1 have center (x, x). The two circles touch at the point P. Using the larger circle, the coordinates of P are $(1-\sqrt{2}/2, 1-\sqrt{2}/2)$. Using C_1, the coordinates of P are $(x+x\sqrt{2}/2, x+x\sqrt{2}/2)$. Solving for x gives

$$x = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

as the radius of C_1.

By similarity, the C_2 has radius x^2, C_3 has radius x^3, and in general, C_n has radius x^n. Then the total area of $C_0, C_1, C_2, \ldots$ is given by

$$\pi(1 + x^2 + x^4 + x^6 + \cdots) = \pi\left(\frac{1}{1-x^2}\right) = \pi\left(\frac{3\sqrt{2}}{8} + \frac{1}{2}\right)$$

(with a little work). Finally, the shaded area is computed as the area of the unit square, plus three fourths of the large circle, minus the total areas of $C_0, C_1, C_2, \ldots$. So, the area is

$$1 + \frac{3\pi}{4} - \pi\left(\frac{3\sqrt{2}}{8} + \frac{1}{2}\right) = 1 + \frac{\pi}{4} - \frac{3\pi\sqrt{2}}{8}.$$

□