SLU Math Team 2009 Qualifying Problems

Do as much as you can, and return your work to Dr. Clair on or before Tuesday, March 31.

1. A $4 \times 4 \times 4$ cube is made from 32 white unit cubes and 32 black unit cubes. What is the largest possible fraction of the surface area that can be black?

   Solution. Use the black cubes for the 8 corners and $12 \times 2 = 24$ edges. Then each face is 3/4 black, so the surface is 3/4 black. (UNCC Math Contest, 1997 # 14). $\square$

2. The function f satisfies $f(0) = 2009$ and has the property that the tangent line to f at x crosses the x-axis at $x + 2009$. Find $f(x)$.

   Solution. The tangent line at p is $l(x) = f'(p)(x - p) + f(p)$. Plugging in $(x + 2009, 0)$ gives $0 = 2009f'(p) + f(p)$, so $f'(x) = -f(x)/2009$. The solution to this differential equation is $f(x) = ke^{-x/2009}$. Using $f(0) = 2009$, we see that $k = 2009$ and $f(x) = 2009e^{-x/2009}$. $\square$

3. Suppose a, b, and c are integers, and suppose $ax^2 + bx + c = 0$ has a rational solution. Prove that at least one of the coefficients a, b, and c must be even.

   Solution. Let $x = p/q$ in lowest terms, so that at least one of p and q is odd. Then
   $$a(p/q)^2 + b(p/q) + c = 0$$
   so
   $$ap^2 + bpq + cq^2 = 0.$$
   If p and q are both odd, then at least one of a, b, and c is even or else $ap^2 + bpq + cq^2$ is the sum of three odd numbers and cannot be zero. If p is odd but q is even, then $bpq + cq^2$ is even, so ap^2 must be even, so a is even. If q is odd, but p is even, then $ap^2 + bpq$ is even, so cq^2 must be even, so c is even. Using modular arithmetic (mod 2) is slightly cleaner. (UIUC Undergraduate Math Contest, 2004) $\square$

4. Equilateral triangles whose side lengths are $1, 3, 5, 7, \ldots$ are placed so that their bases lie corner to corner along a straight line. Show that the vertices lie along a parabola.

   Solution. Assume without loss of generality the bases lie on the positive x-axis with the first corner at the origin. Then the vertices are at $(1/2, \sqrt{3}/2), (5/2, 3\sqrt{3}/2), (13/2, 5\sqrt{3}/2), \ldots, (k^2-k+1/2, (2k+1)\sqrt{3}/2), \ldots$. Setting $x = ay^2 + c$, plugging in points, and solving for a and c, we find that these points all satisfy $x = \frac{1}{3}y^2 + \frac{1}{4}$. (Larson 8.2.8) $\square$

5. Prove

$$\int_0^1 \frac{dx}{x^x} = \sum_{n=1}^{\infty} \frac{1}{n^n}$$

Solution.

$$\begin{aligned}
\int_0^1 \frac{dx}{x^x} &= \int_0^1 e^{-x\log x} dx && (1)\\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x\log x)^n}{n!} dx && (2)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \log^n(x) dx. && (3)
\end{aligned}$$

The interchange of sum and integral is justified by uniform convergence. Let $S_{j,k}(x) = x^j \log^k(x)$. Claim that

$$\int_0^1 S_{j,k}(x)dx = \frac{(-1)^k k!}{(j+1)^{k+1}}.$$

The proof is by induction on k, with $k = 0$ clear by the power rule. Using integration by parts,

$$\begin{aligned}
\int_0^1 S_{j,k}(x)dx &= \frac{1}{j+1} S_{j+1,k}(x)\Big]_0^1 - \frac{k}{j+1}\int_0^1 S_{j,k-1}(x)dx && (4)\\
&= -\frac{k}{j+1}\int_0^1 S_{j,k-1}(x)dx && (5)\\
&= -\frac{k}{j+1}\frac{(-1)^{k-1}(k-1)!}{(j+1)^k} && (6)\\
&= \frac{(-1)^k k!}{(j+1)^k+1}, && (7)
\end{aligned}$$

using $\lim_{x\to 0^+} S_{j,k}(x) = 0$ for $j > 0$ and all k. This proves the claim.

Finally,

$$\int_0^1 \frac{dx}{x^x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 S_{n,n}(x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

(Amer. Math Monthly, 59:2 pp 108-109, see Bonar & Khoury, Real Infinite Series, Gem 30) □