SLU Math Team 2009 Qualifying Problems

Do as much as you can, and return your work to Dr. Clair on or before Tuesday, March 31.

1. A $4 \times 4 \times 4$ cube is made from 32 white unit cubes and 32 black unit cubes. What is the largest possible fraction of the surface area that can be black?

2. The function f satisfies $f(0) = 2009$ and has the property that the tangent line to f at x crosses the x-axis at $x + 2009$. Find $f(x)$.

3. Suppose a, b, and c are integers, and suppose $ax^2 + bx + c = 0$ has a rational solution. Prove that at least one of the coefficients a, b, and c must be even.

4. Equilateral triangles whose side lengths are $1, 3, 5, 7, \ldots$ are placed so that their bases lie corner to corner along a straight line. Show that the vertices lie along a parabola.

5. Prove
$$\int_0^1 \frac{dx}{x^x} = \sum_{n=1}^{\infty} \frac{1}{n^n}$$