SLU Math Team 2008 Qualifying Problems

Do as much as you can, and return your work to Dr. Clair on or before Monday, April 7.

1. Four spheres of radius 1 are stacked so that each is tangent (externally) to the other three. What is the radius of the largest sphere that can fit into the space between them?

   Solution. The radius is $\sqrt{6}/2 - 1$. The centers of the four spheres are the vertices of a regular tetrahedron with edge length 2. Call the top vertex T, and any one of the lower three O. Denote by P the center of the small sphere that fits in the space. From P, drop a perpendicular to the bottom face of the tetrahedron, hitting at a point Q. Then it's not hard to show that $OQ = \frac{2}{\sqrt{3}}$. The Pythagorean theorem gives $PQ^2 + OQ^2 = OP^2$, so

   $$PQ^2 + \frac{4}{3} = OP^2. \tag{1}$$

   Now notice $TQ = TP + PQ = OP + PQ$, and $TO = 2$, so the Pythagorean theorem on triangle TOQ gives

   $$(OP + PQ)^2 + \frac{4}{3} = 4. \tag{2}$$

   Solving (1) and (2) gives $OP = \sqrt{6}/2$. The radius is $OP - 1$. □

2. The Missouri Lotto drawing fills a bin with balls numbered 1-49, then draws five balls, one at a time without replacement. What is the probability that the balls are drawn in increasing numerical order?

   Solution. 1/120. Whatever five balls are drawn, there are $5! = 120$ possible orders they can have. All orders are equally likely, and only one is increasing. Source: MIT 18.S34 (Fall 2007) problems on independence and uniformity. □

3. Determine whether the improper integral

   $$\int_0^\infty (-1)^{\lfloor x^2 \rfloor} dx$$

   converges or diverges, where $\lfloor \cdot \rfloor$ is the greatest integer function.

   Solution. It converges. For $\sqrt{n} \le x < \sqrt{n+1}$, we have $\lfloor x^2 \rfloor = n$, so that

   $$\int_0^\infty (-1)^{\lfloor x^2 \rfloor} dx = \sum_{n=0}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}).$$

   Since the series is alternating and its terms approach zero, it converges. Source: Youngstown State Calculus Competition, 2005. □

4. Let f and g be functions from the set $\mathbb{R}$ of real numbers to itself, such that $g(x) < f(x)$ for all $x \in \mathbb{R}$. Prove there exists an infinite subset $S \subseteq \mathbb{R}$ such that $g(x) < f(y)$ for all $x, y \in S$.

   Solution. Let $S_r = \{x | g(x) < r < f(x)\}$. Then S_r satisfies $g(x) < f(y)$ for any $x, y \in S_r$. For any x, there is some rational number r so that $x \in S_r$. Then $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} S_r$. Since $\mathbb{R}$ is not countable, it's not the countable union of finite sets, and so some S_r must be infinite (in fact, uncountable). Question: is this statement still true if f and g are only defined on $\mathbb{Q}$?

   Source: Berkeley Math Problem Circle Monthly Contest, Nov. 2000. □

5. If f is a polynomial of degree n such that $f(i) = 2^i$ for $i = 0, 1, \ldots, n$, find $f(n+1)$.

   Solution. $f(n+1) = 2^{n+1} - 1$. There's probably a better way than this solution, but it's what I came up with. We need the following fact:

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} f(k) = 0 \tag{3}$$

   for all polynomials of degree less than m.

   Now, let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$. Then using (3) with $m = n+1$,

$$\begin{aligned} 0 &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} f(k) && (4) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} 2^k + (-1)^{n+1} f(n+1) && (5) \\ &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} 2^k - (-1)^{n+1} 2^{n+1} + (-1)^{n+1} f(n+1) && (6) \\ &= (1 + (-2))^{n+1} - (-1)^{n+1} 2^{n+1} + (-1)^{n+1} f(n+1), && (7) \end{aligned}$$

   so that $0 = 1 - 2^{n+1} + f(n+1)$, or $f(n+1) = 2^{n+1} - 1$.

   We now prove (3). We only need to show that, for $p = 0, 1, \ldots, m-1$, we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} k^p = 0 \tag{8}$$

   The case $p = 0$ is easy, since $(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$ and putting $x = -1$ gives the result. The case $m = 0$ is even easier. We now induct on m, i.e. assume that (8) is true for $m-1$ and for all $0 \leq p < m-1$. Now suppose

$0 < p < m$.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^p = \sum_{k=1}^{m}(-1)^k\binom{m-1}{k-1}\frac{m}{k}k^p \tag{9}$$

$$= -n\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(k+1)^{p-1} \tag{10}$$

$$= -n\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\left(k^{p-1}+\binom{p-1}{1}k^{p-2}+\ldots+\binom{p-1}{p-2}k+1\right) \tag{11}$$

$$= 0, \tag{12}$$

where the last step used the induction hypothesis for every term $k^{p-1}, \ldots, k^1, k^0$.

From 1998 IMO proposal (via G.D. Carroll, Berkeley Math Circle). $\square$