SLU Math Team 2007 Qualifying Problems

Do as much as you can, and return your work to Dr. Clair on or before Tuesday, March 20.

1. Prove that

$$\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}$$

exists and find its exact value.

Solution. Let $x_1 = \sqrt{3}$, $x_2 = \sqrt{3+\sqrt{3}}$, and generally $x_n = \sqrt{3+x_{n-1}}$. The value we want is $lim_{n\to\infty} x_n$, if it exists.

Assume for the moment the value exists, and call it x. Then $x = \sqrt{3+x}$ and so $x^2 - x - 3 = 0$. Then $x = (1 \pm \sqrt{13})/2$ and clearly x is positive. So $x = (1+\sqrt{13})/2$.

Now we show the limit exists. Clearly $x_1 < x_2$ and assume that $x_{n-1} < x_n$ $(n > 1)$. Then

$$x_{n+1} = \sqrt{3+x_n} > \sqrt{3+x_{n-1}} = x_n,$$

and by induction, the sequence $x_1, x_2, \ldots$ is (strictly) increasing. Next, note $x_1 < 3$, and assume that $x_n < 3$. Since

$$x_{n+1} = \sqrt{3+x_n} < \sqrt{3+3} < 3,$$

by induction the entire sequence is bounded above by 3. Any increasing sequence bounded above must have a limit. □

2. SLU's box at Busch stadium has six seats arranged in two rows of three. If Father Biondi and five rich alumni go to a game, how many ways can the six people sit so that each person in the front row is shorter than the person directly behind them in the back row. Assume all six people are different heights.

Solution. Generalizing this problem a bit, suppose there are n seats in front and n in back, with $2n$ people of differing heights. Let S_n be the number of ways to seat $2n$ people of different heights so that each person in the front row is shorter than the person directly behind them in the back row.

The shortest person must sit in the front row, and has n choices of seat. Any of the $2n-1$ remaining people may sit behind the shortest person. This leaves $2n-2$ people to seat in the remaining $n-1$ seats, and there are S_{n-1} ways to do this. Thus,

$$S_n = n(2n-1)S_{n-1}.$$

Also, $S_1 = 1$ since with 2 people and 2 seats, the shorter person must take the front seat. Now a simple induction argument shows that

$$S_n = n!(2n-1)(2n-3)\cdots 3 \cdot 1.$$

In particular, the problem asks for $S_3 = 90$.

Source: KoMaL, March 2005, K40. □

3. Let $\alpha \geq 1$ and let $f(x) = x^\alpha$. Let $b(x)$ be the y-intercept of the normal line to f at $(x, f(x))$. What is $\lim_{x\to 0+} b(x)$?

   Solution. The tangent line at $x = a$ has slope $\alpha a^{\alpha-1}$ so the normal line has slope $-\frac{1}{\alpha a^{\alpha-1}}$ and equation

   $$y - a^\alpha = -\frac{1}{\alpha a^{\alpha-1}}(x-a)$$

   so that $b(a) = \frac{1}{\alpha a^{\alpha-2}} + a^\alpha$. Since $a > 1$, $a^\alpha \to 0$ as $a \to 0$. Then

   $$\lim_{a\to 0} b(a) = \begin{cases} 0 & \alpha < 2, \\ 1/2 & \alpha = 2, \\ \infty & \alpha > 2 \end{cases}$$

   Source: Alberto Delgado, Bradley U. Problem of the Week #200. □

4. Let $B_1 = 0$ and $B_2 = 1$. For $n > 2$, the number B_n is defined by writing the decimal digits of B_{n-1} followed by the digits of B_{n-2}. For example $B_3 = B_2B_1 = 10$, $B_4 = B_3B_2 = 101$ and $B_5 = B_4B_3 = 10110$. Determine all n so that 11 divides B_n.

   Solution. $n = 6k+1$ for $k = 0, 1, 2, \ldots$.

   Let L_n denote the length of the decimal expansion of B_n. Since $L_1 = L_2 = 1$ and $L_n = L_{n-1} + L_{n-2}$, the L_n are just the Fibonacci numbers and it is not hard to see that L_n is even if and only if $3|n$.

   Now $B_n = 10^{L_{n-2}}B_{n-1} + B_{n-2}$. Since $10 \equiv -1 \pmod{11}$, we have $B_n \equiv B_{n-1} + B_{n-2} \pmod{11}$ when $3|n$ and $B_n \equiv -B_{n-1} + B_{n-2} \pmod{11}$ otherwise.

   Computing mod 11, the sequence $B_1, B_2, \ldots$ begins $0, 1, -1, 2, 1, 1, 0, 1, -1$. Since the sign to generate the next term has period 3 the sequence will repeat $0, 1, -1, 2, 1, 1$ (with period 6) forever.

   Source: Ravi Vakil Putnam Seminar 2004, Number Theory #20. □

5. Let T be the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$ for some $a,b,c>0$. Let V be the volume of T and let ℓ be the sum of the lengths of the six edges of T. Prove that

$$V \le \frac{\ell^3}{6(3+3\sqrt{2})^3}$$

Solution. First, using the Pythagorean theorem,

$$\ell = a+b+c+\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}.$$

The AM-GM inequality gives:

$$a+b+c \ge 3\sqrt[3]{abc}.$$

Using AM-GM again, and then once more:

$$\begin{aligned}
&\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\\
&\qquad \ge 3\sqrt[3]{\sqrt{a^2+b^2}\sqrt{b^2+c^2}\sqrt{c^2+a^2}}\\
&\qquad = 3\sqrt[6]{a^2b^2c^2+a^4b^2+a^2c^4+a^4c^2+b^4c^2+b^4a^2+b^2c^4+a^2b^2c^2}\\
&\qquad \ge 3\sqrt[6]{8\sqrt[8]{a^{16}b^{16}c^{16}}}\\
&\qquad = 3\sqrt[6]{8a^2b^2c^2}\\
&\qquad = 3\sqrt{2}\sqrt[3]{abc}.
\end{aligned}$$

Putting it together,

$$\ell \ge 3\sqrt[3]{abc}+3\sqrt{2}\sqrt[3]{abc} = (3+3\sqrt{2})\sqrt[3]{abc}.$$

Then

$$\frac{\ell^3}{6(3+3\sqrt{2})^3} \ge \frac{abc}{6} = V.$$

Source: Bjorn Poonen's Inequalities #10. □