Series Problems

1. Sum the infinite series

$$\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\cdots$$

Solution. $5e$. Let

$$f(x)=xe^x=\frac{x}{0!}+\frac{x^2}{1!}+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots$$

Then

$$xf'(x)=xe^x+x^2e^x=\frac{x}{0!}+\frac{2x^2}{1!}+\frac{3x^3}{2!}+\frac{4x^4}{3!}+\cdots$$

So

$$(xf'(x))'=e^x+3xe^x+x^2e^x=\frac{1}{0!}+\frac{2^2x}{1!}+\frac{3^2x^2}{2!}+\frac{4^2x^3}{3!}+\cdots$$

Plugging in $x=1$ gives

$$5e=\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\cdots$$

□

2. Sum the infinite series

$$\sum_{i=1}^{\infty}\frac{1}{(3i-2)(3i+1)}$$

Solution. The sum is 1/3. Using partial fractions,

$$\frac{1}{(3i-2)(3i+1)}=\frac{1}{3}\left(\frac{1}{3i-2}-\frac{1}{3i+1}\right)$$

Then the N^{th} partial sum telescopes to equal $\frac{1}{3}(1-1/(3N+1))$, which has limit $1/3$ as $N\to\infty$. □

3. (MCMC 2008I #5) Evaluate

$$\sum_{k=1}^{\infty}\frac{1}{\binom{k+n}{k}}$$

for $n\geq 2$. What is this series when $n=1$?

Solution.

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}} = \sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{n}} = \sum_{k=1}^{\infty} \frac{1}{\frac{(k+1)(k+2)\cdots(k+n)}{n!}} \\
&= n! \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+n)} \\
&= \frac{n!}{n-1} \sum_{k=1}^{\infty} \left(\frac{1}{(k+1)(k+2)\cdots(k+n-1)} - \frac{1}{(k+2)(k+3)\cdots(k+n)} \right) \\
&= \frac{n!}{n-1} \left(\frac{1}{2\cdot 3 \cdot \cdots \cdot n} - \frac{1}{3 \cdot 4 \cdot \cdots \cdot (n+1)} + \frac{1}{3 \cdot 4 \cdot \cdots \cdot (n+1)} - \frac{1}{4 \cdot 5 \cdot \cdots \cdot (n+2)} \right. \\
&\qquad \left. + \frac{1}{4 \cdot 5 \cdot \cdots \cdot (n+2)} - \frac{1}{5 \cdot 6 \cdot \cdots \cdot (n+3)} + \cdots \right) \\
&= \frac{n!}{n-1} \cdot \frac{1}{n!} = \frac{1}{n-1}. \qquad (1)
\end{aligned}$$

For $n = 1$, the series is a harmonic series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which is divergent, and the formula $1/(n-1)$ would indicate that the series should be divergent. □

4. (MCMC 2009I#4) Find the value of the infinite product

$$\left(\frac{7}{9}\right) \cdot \left(\frac{26}{28}\right) \cdot \left(\frac{63}{65}\right) \cdot \cdots = \lim_{n\to\infty} \prod_{k=2}^{n} \left(\frac{k^3-1}{k^3+1}\right).$$

Solution. We rewrite the nth partial product so as to reveal two sets of telescoping products:

$$\begin{aligned}
\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} &= \prod_{k=2}^{n} \left(\frac{k-1}{k+1}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{k^2-k+1}\right) && (2) \\
&= \prod_{k=2}^{n} \left(\frac{k-1}{k+1}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right) && (3) \\
&= \prod_{k=2}^{n} \left(\frac{(k-1)((k-1)+1)}{k(k+1)}\right) \prod_{k=2}^{n} \left(\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right) && (4) \\
&= \frac{2}{n(n+1)} \cdot \frac{n^2+n+1}{3} && (5) \\
&= \frac{2}{3}\left(1 + \frac{1}{n(n+1)}\right). && (6)
\end{aligned}$$

Hence,

$$\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1} = \frac{2}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n(n+1)}\right) = \frac{2}{3}.$$

□

5. (MCMC 2011#5) Evaluate the series

$$\sum_{n=0}^{\infty} \frac{1}{2011^{2^n} - 2011^{-2^n}} = \frac{1}{2011^1 - 2011^{-1}} + \frac{1}{2011^2 - 2011^{-2}} + \frac{1}{2011^4 - 2011^{-4}} + \cdots$$

and express it as a rational number.

Solution. Let

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}, \quad S_N(x) = \sum_{n=0}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

The series in the problem is $S(1/2011)$. First, check the following identity:

$$\frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}}.$$

Using this identity, the partial sums telescope:

$$\begin{aligned} S_N(x) &= \left(\frac{1}{1-x} - \frac{1}{1-x^2}\right) + \left(\frac{1}{1-x^2} - \frac{1}{1-x^4}\right) + \cdots + \left(\frac{1}{1-x^{2^N}} - \frac{1}{1-x^{2^{N+1}}}\right) \\ &= \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}} \end{aligned}$$

so that

$$S(x) = \lim_{N\to\infty} S_N(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

Finally, $S(1/2011) = (1/2011)/(1 - 1/2011) = 1/2010$. □

6. Let p and q be real numbers with $1/p - 1/q = 1$, $0 < p \leq \frac{1}{2}$. Show that

$$p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \cdots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \cdots$$

Solution. First, note that $0 < p \leq \frac{1}{2}$ implies $0 < q \leq 1$. So, when $p < 1/2$, $q < 1$ the two series sum (by standard power series) to $-\log(1-p)$ and $\log(1+q)$.

Manipulating $1/p - 1/q = 1$ gives $1 - p = p/q$ and $1 + q = q/p$, and so

$$-\log(1-p) = -\log(p/q) = \log(q/p) = \log(1+q)$$

.

When $p = 1/2$, the series in p still sums to $-\log(1-p) = \log 2$. Here, $q = 1$ which is outside the radius of convergence of $\log(1+q)$. However,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$$

is well known (it follows from Abel's Limit Theorem). □

7. Repeatedly toss a fair coin. What is the probability that the first head occurs on an even-numbered toss?

Solution. It is $\frac{1}{3}$. The first head occurs on toss n if there are $n-1$ tails followed by a head. This has probability $(\frac{1}{2})^{n-1} \cdot \frac{1}{2} = \frac{1}{2^n}$. Then the probability the first head occurs on an even numbered toss is

$$\sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}.$$

□

8. Sum the series

$$1 + 22 + 333 + \cdots + n(\overbrace{11\ldots1}^{n})$$

Solution. Let $A_n = 1 + 22 + 333 + \cdots + n(11\ldots1)$. Then

$$A_n = \frac{10^{n+1}(9n-1)}{9^3} + \frac{10}{9^3} - \frac{n(n+1)}{18}.$$

Define $S_n(x) = x + 2x^2 + 3x^3 + \cdots nx^n$. Then

$$\begin{aligned} 9A_n &= 9 + 2\cdot 99 + 3\cdot 999 + \cdots + n(99\ldots99) \\ &= (10-1) + 2(10^2-1) + 3(10^3-1) + \cdots + n(10^n-1) \\ &= 10 + 2\cdot 10^2 + 3\cdot 10^3 + \cdots + n\cdot 10^n - (1+2+3+\cdots+n) \\ &= S_n(10) - \frac{n(n+1)}{2}. \end{aligned}$$

Now

$$\begin{aligned} \frac{S_n(x)}{x} &= 1 + 2x + 3x^2 + \cdots nx^{n-1} \\ &= \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots + x^n) \\ &= \frac{d}{dx}\left(\frac{x^{n+1}-1}{x-1}\right) \\ &= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}. \end{aligned}$$

Finally,

$$
\begin{aligned}
A_n &= \frac{1}{9}\left(10\left(\frac{n10^{n+1}-(n+1)10^n+1}{9^2}\right)-\frac{n(n+1)}{2}\right)\\
&= \frac{10^{n+1}(9n-1)}{9^3}+\frac{10}{9^3}-\frac{n(n+1)}{18}.
\end{aligned}
$$

□

9. Find the limit as $n \to \infty$ of the sum

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\cdots+\frac{1}{2n}.$$

Solution. The solution is $\log(2)$. The sum S_n satisfies.

$$\int_n^{2n} x^{-1}dx < S_n < \int_{n-1}^{2n-1} x^{-1}dx$$

By direct integration, both integrals have limit $\log(2)$.

□