Series Problems

1. Sum the infinite series
$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

2. Sum the infinite series
$$\sum_{i=1}^{\infty} \frac{1}{(3i-2)(3i+1)}$$

3. (MCMC 2008I #5) Evaluate
$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+n}{k}}$$
for $n \geq 2$. What is this series when $n = 1$?

4. (MCMC 2009I#4) Find the value of the infinite product
$$\left(\frac{7}{9}\right) \cdot \left(\frac{26}{28}\right) \cdot \left(\frac{63}{65}\right) \cdot \cdots = \lim_{n\to\infty} \prod_{k=2}^{n} \left(\frac{k^3-1}{k^3+1}\right).$$

5. (MCMC 2011#5) Evaluate the series
$$\sum_{n=0}^{\infty} \frac{1}{2011^{2^n} - 2011^{-2^n}} = \frac{1}{2011^1 - 2011^{-1}} + \frac{1}{2011^2 - 2011^{-2}} + \frac{1}{2011^4 - 2011^{-4}} + \cdots$$
and express it as a rational number.

6. Let p and q be real numbers with $1/p - 1/q = 1$, $0 < p \leq \frac{1}{2}$. Show that
$$p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \cdots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \cdots$$

7. Repeatedly toss a fair coin. What is the probability that the first head occurs on an even-numbered toss?

8. Sum the series
$$1 + 22 + 333 + \cdots + n(\overbrace{11\ldots1}^{n})$$

9. Find the limit as $n \to \infty$ of the sum
$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \cdots + \frac{1}{2n}.$$