Pigeonhole Principle Problems

These are some solutions to problems from Ravi Vakil's handout.

13. *Solution.* Let S be any set of $n+1$ distinct integers between 1 and $2n$. Define n sets $T_i = \{i, 2i, 4i, 8i, 16i, \dots\}$ with $i = 1, 3, 5, \dots, 2n-1$. The set $T_1 \cup T_3 \cup T_5 \cup \cdots \cup T_{2n-1}$ contains all integers from 1 to $2n$, and therefore contains all $n+1$ elements of S. By the pigeonhole principle, some set T_i contains at least two elements $x, y \in S$. If $x < y$ then $y = 2^k x$ and so x divides y. Similarly if $y < x$, then y divides x. So, one of the elements of S divides another.

    This is the best possible, since none of the n integers $n+1, n+2, \dots, 2n$ divide each other. □

14. *Solution.* Dissect the triangle into four smaller equilateral triangles: $\overset{\triangle}{\triangle\triangle}$. Two of the five points must be in one of the smaller triangles. Each of these has side length $1/2$, so the two points are within $1/2$ of each other. □

16. *Solution.* A six-inch radius sphere has area $4\pi 6^2 \approx 450$ square inches, which would be a really huge area for a human head. If you can fit 100 hairs along one linear inch, you can get 10000 hairs in a square inch (again, a vast overestimate). This means that a human head has less (probably much less) than 4,500,000 hairs. It seems reasonable that of the more than 8 million people in New York, there are at least 4.5 million who are not bald. So, we have more than 4.5 million hirsute people, and at most 4.5 million possible hair counts. The pigeonhole principle implies that two people have the same hair count. □

17. *Solution.* Consider the pairs $(4, 100), (7, 97), (10, 94), \dots, (49, 55)$. There are sixteen such pairs. The set A may contain 1 and 52, but has at least 18 other numbers which must be contained in these sixteen pairs. By the pigeonhole principle, one pair must contain two numbers from A, and those two numbers add to 104. □

18. *Solution.* We assume that knowing is a symmetric relation: If person A knows person B, then person B knows person A. Without this assumption, the problem is false, since we may have a party with two people where A knows zero people and B knows one person, A.

    We also make the assumption that knowing yourself does not count, and 'knowing people' means knowing *other* people. This assumption is unimportant, because the alternative simply adds 1 to all the numbers.

    Let K_i be the number of people that person i knows. There are apparently n possibilities for K_i, since a person may know $0, 1, \dots, n-1$ other people. However, it is not possible to have a party where one person (the stranger) knows 0 people and someone else knows $n-1$ people (the host): To know $n-1$ people, the host would have to know the stranger, but the stranger does not know the host.

Therefore, there are only $n-1$ possibilities for K_i, and there are n numbers $K_1, \dots, K_n$. Two of these must be the same. □

24. *Solution.* For an invertible matrix with integer entries, the inverse matrix has invertible entries if and only if the determinant of the matrix is ± 1 (this follows from Cramer's Rule).

    Let $p(k) = \det(A+kB)$. p is a quadratic polynomial in k. The hypothesis say that $p(0), p(1), p(2), p(3), p(4) \in \{\pm 1\}$. By the pigeonhole principle, there must be distinct $k_1, k_2, k_3 \in \{0, 1, 2, 3, 4\}$ with $p(k_1) = p(k_2) = p(k_3) = s$, where $s = \pm 1$. A quadratic polynomial which takes the same value at three distinct points must be constant (consider $p(k) - s$ which has three roots). So

    $$\det(A + kB) = p(k) = s$$

    for all k, and in particular, $\det(A + 5B) = s = \pm 1$. □

25. *Solution.* This is problem A3 from the 1990 Putnam exam. There are solutions online. □