Number Theory and Modular Arithmetic Problems

1. Suppose a prime number p is the sum of two squares. Show that $p-1$ is divisible by four.

2. Determine all integral solutions of $a^2+b^2+c^2=a^2b^2$. (Hint: Work modulo 4).

3. Prove that for any set of n integers, there is a subset of them whose sum is divisible by n.

4. Suppose $f(x)$ is a polynomial with integral coefficients, and none of the integers $f(1), f(2), \ldots, f(2007)$ is divisible by 2007. Prove that f has no integral zero.

5. A *lattice point* is a point whose coordinates are both integers. If a and b and chosen randomly from $1, 2, \ldots, 100$. What is the probability that the segment from $(0, a)$ to $(b, 0)$ contains an even number of lattice points?

6. Find the largest integer that is equal to the product of its digits.

7. Imagine you are at a school that has 100 lockers, all shut. Suppose the first student goes along the row and opens every locker. The second student then goes along and shuts every other locker beginning with locker number 2. The third student changes the state of every third locker beginning with locker number 3. (If the locker is open the student shuts it, and if the locker is closed the student opens it.) The fourth student changes the state of every fourth locker beginning with number 4. Imagine that this continues until the 100 students have followed the pattern with the 100 lockers. At the end, which lockers will be open and which will be closed?

8. Do there exist 100 consecutive integers so that each is divisible by a perfect cube bigger than 1?

9. Show that the sequence $11, 111, 1111, 11111, \ldots$ contains no perfect squares.

10. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits of infinitely many a_i?