L'Hopital's Rule

Theorem (L'Hopital's rule)**.** *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$*, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)},$$

provided the derivatives in question exist for all $x \neq a$ *and provided the right hand limit exists.*

Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for x.

Problems

1. Evaluate

$$\lim_{n\to\infty} 4^n \left(1 - \cos(\frac{\theta}{2^n})\right),$$

Solution. Let $x = 2^{-n}$. Then the limit becomes

$$\begin{aligned}
\lim_{n\to\infty} 4^n \left(1 - \cos(\frac{\theta}{2^n})\right) &= \lim_{x\to 0^+} \frac{1-\cos(\theta x)}{x^2} && (1)\\
&= \lim_{x\to 0^+} \frac{\theta \sin(\theta x)}{2x} \quad \text{(L'Hopital's)} && (2)\\
&= \lim_{x\to 0^+} \frac{\theta^2 \cos(\theta x)}{2} \quad \text{(L'Hopital's)} && (3)\\
&= \theta^2/2. && (4)
\end{aligned}$$

□

2. Evaluate the following limits:

(a)

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Solution. e □

(b)

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$

Solution. e □

(c)

$$\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$$

Solution. 1 □

(d)

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}$$

Solution. ∞ □

3. Evaluate
$$\lim_{n\to\infty}\frac{2p_nP_n}{p_n+P_n},$$
where $p_n=(1+1/n)^n$ and $P_n=(1+1/n)^{n+1}$.

Solution. e □

4. Evaluate
$$\lim_{x\to\infty}\left(\frac{1}{x}\frac{a^x-1}{a-1}\right)^{1/x}.$$
where $a>1$.

Solution.
$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{1}{x}\frac{a^x-1}{a-1}\right)^{1/x} &= \lim_{x\to\infty}\left(\frac{1}{x}a^x\frac{1-a^{-x}}{a-1}\right)^{1/x}\\
&= \lim_{x\to\infty}\left(\frac{1}{x}\right)^{1/x} a\left(\frac{1-a^{-x}}{a-1}\right)^{1/x}\\
&= a\lim_{x\to\infty}\left(\frac{1}{x}\right)^{1/x}\lim_{x\to\infty}\left(\frac{1-a^{-x}}{a-1}\right)^{1/x}\\
&= a\left(\frac{1}{a-1}\right)^{0}\lim_{x\to\infty}\exp\left(\frac{1}{x}\ln\frac{1}{x}\right)\\
&= a\exp\left(\lim_{x\to\infty}\frac{-\ln x}{x}\right)\\
&= a\exp(0)=a.
\end{aligned}$$

□

5. Let $f(t)$ and $f'(t)$ be differentiable on $[a,x]$ and for each x suppose there is a number c_x such that $a<c_x<x$ and
$$\int_a^x f(t)dt = f(c_x)(x-a).$$
Assume that $f'(a)\neq 0$. Then prove that
$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

Solution. This is from the MCMC 2006 Session I, problem 5. See their solution. □

6. Calculate
$$\lim_{x\to\infty} x \int_0^x e^{t^2-x^2}\,dt.$$

Solution.
$$x\int_0^x e^{t^2-x^2}\,dt = \frac{x\int_0^x e^{t^2}\,dt}{e^{x^2}}$$
Now the limit is expressed as an indeterminate form $\frac{\infty}{\infty}$, so apply L'Hopital's Rule, the product rule, and the fundamental theorem of calculus, to get:
$$\begin{aligned}
\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt &= \lim_{x\to\infty} \frac{\int_0^x e^{t^2}\,dt + xe^{x^2}}{2xe^{x^2}} \\
&= \lim_{x\to\infty} \frac{\int_0^x e^{t^2}\,dt}{2xe^{x^2}} + \frac{1}{2} \\
&= \lim_{x\to\infty} \frac{e^{x^2}}{2e^{x^2} + 4x^2e^{x^2}} + \frac{1}{2} \quad \text{(L'Hopital's again)} \\
&= \lim_{x\to\infty} \frac{1}{2+4x^2} + \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$
□

7. Prove that the function $y = (x^2)^x$, $y(0) = 1$, is continuous at $x = 0$.

Solution.
$$\begin{aligned}
\lim_{x\to 0}(x^2)^x &= \lim_{x\to 0}\exp(x\ln x^2) \\
&= \exp\left(\lim_{x\to 0}\frac{\ln x^2}{x^{-1}}\right) \\
&= \exp\left(\lim_{x\to 0}\frac{\frac{1}{x^2}2x}{-x^{-2}}\right) \\
&= \exp\left(\lim_{x\to 0} -2x\right) \\
&= 1
\end{aligned}$$
Which shows that $y \to y(0)$ as $x \to 0$, so y is continuous at 0. □