L'Hopital's Rule

Theorem (L'Hopital's rule)**.** *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$*, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)},$$

provided the derivatives in question exist for all $x \neq a$ *and provided the right hand limit exists.*

Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for x.

Problems

1. Evaluate
$$\lim_{n\to\infty} 4^n \left(1 - \cos(\frac{\theta}{2^n})\right),$$
2. Evaluate the following limits:
   (a)
   $$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$
   (b)
   $$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$
   (c)
   $$\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$$
   (d)
   $$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$
3. Evaluate
$$\lim_{n\to\infty} \frac{2p_n P_n}{p_n + P_n},$$
where $p_n = (1 + 1/n)^n$ and $P_n = (1 + 1/n)^{n+1}$.
4. Evaluate
$$\lim_{x\to\infty} \left(\frac{1}{x}\frac{a^x - 1}{a - 1}\right)^{1/x}.$$
where $a > 1$.

5. Let $f(t)$ and $f'(t)$ be differentiable on $[a, x]$ and for each x suppose there is a number c_x such that $a < c_x < x$ and

$$\int_a^x f(t)dt = f(c_x)(x - a).$$

Assume that $f'(a) \neq 0$. Then prove that

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

6. Calculate

$$\lim_{x \to \infty} x \int_0^x e^{t^2 - x^2} dt.$$

7. Prove that the function $y = (x^2)^x$, $y(0) = 1$, is continuous at $x = 0$.