Mathematical Induction

The *natural numbers* are the counting numbers: $1, 2, 3, 4, \ldots$. *Mathematical induction* is a technique for proving a statement - a theorem, or a formula - that is asserted about every natural number. For example,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

This asserts that the sum of consecutive numbers from 1 to n is given by the formula on the right. We want to prove that this will be true for $n = 1$, $n = 2$, $n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$:

$$1 + 2 + 3 = \frac{3 \times 4}{2} = 6,$$

which is true. It is also true for $n = 4$:

$$1 + 2 + 3 + 4 = \frac{4 \times 5}{2} = 10.$$

But how are we to prove this rule for *every* value of n? The method of proof is the following:

Principle of Mathematical Induction.
Suppose

1) (The base case) The statement is true for $n = 1$;

2) If the statement is true for n, then it is also true for $n + 1$;

Then the statement is true for every natural number n.

When the statement is true for $n = 1$, then according to 2), it will also be true for $n = 2$. But that implies it will be true for $n = 3$; which implies it will be true for $n = 4$. And so on. It will be true for every natural number. To prove a statement by induction, then, we must prove parts 1) and 2) above.

The hypothesis of part 2) - "The statement is true for n" - is called the inductive assumption, or the inductive hypothesis. It is what we assume when we prove a theorem by induction.

Example 1. Show that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{1}$$

Proof. For $n = 1$, we have $1 = \frac{1(1+1)}{2}$ which is true.

Suppose (the induction hypothesis) that the statement (1) is true for n:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Then

$$
\begin{aligned}
1+2+3+\ldots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\
&= \frac{n^2+n+2n+2}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2},
\end{aligned}
$$

which proves the statement (1) for $n+1$. By induction, the statement (1) is true for all natural numbers n. □

For the base case of induction, it is not necessary to use $n = 1$. Any other base number k will work, and the result of induction will be that the statement is true for any $n \geq k$.

There is also a technique called *strong induction*, in which the inductive hypothesis is that the statement is true for $1, 2, 3, \ldots, n$.

Problems

1. Prove that $n! > 2^n$ for all $n \geq 4$.

   Solution. When $n = 4$,

   $$4! = 24 > 16 = 2^4,$$

   so the statement is true. Assume $n! > 2^n$. Then

   $$(n+1)! = (n+1)n! > (n+1)2^n > 2 \cdot 2^n = 2^{n+1}$$

   (where we use the fact that $n+1 > 2$). By induction, $n! > 2^n$ for all $n \geq 4$. □

2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

3. Prove that all numbers in the sequence $1007, 10017, 100117, 1001117, 10011117, \ldots$ are divisible by 53.

   Solution. Let $a_n = 100111\cdots117$ where there are n 1's. Check $a_0 = 1007 = 53 * 19$. Now suppose a_n is divisible by 53. Generally, $a_{n+1} = ((a_n - 7) + 1) * 10 + 7 = 10a_n - 53$. Since both 53 and a_n are divisible by 53, so a_{n+1} is as well. □

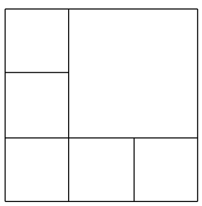
4. Let F_k be the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$. Show that:

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

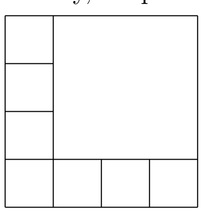
5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer.

6. Prove that any square can be dissected into n smaller squares (possibly of differing sizes) for every $n \geq 6$.

   Solution. First, if you can dissect a square into n squares, then you can dissect into $n + 3$ squares as follows: Choose any square in the dissection, and replace it with four squares, each one quarter of the original square. Since a square can be dissected into one square, induction proves that a square can be dissected into $1, 4, 7, 10, 13, \ldots$ squares. A square can be dissected into six squares as follows:



   By induction, then, a square can be dissected into $6, 9, 12, 15, 18, \ldots$ squares. Finally, a square can be dissected into eight squares as follows:



   By induction, a square can be dissected into $8, 11, 14, 17, \ldots$. In summary a square can be dissected into $1, 4, 6, 7, 8, 9, 10, \ldots$ squares, a list which includes every number greater than or equal to six. □

7. Show that:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2 \cos\left(\frac{\pi}{2^{n+1}}\right),$$

   where there are n 2s in the expression on the left.

8. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only.

9. Prove *Bernoulli's Inequality*:

$$(1+x)^n \geq 1 + nx$$

for every real number $x \geq -1$ and every natural number n.

10. Prove that $2^{2^n} + 3^{2^n} + 5^{2^n}$ is divisible by 19 for all positive integers n.

11. Prove that $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer for $n = 0, 1, 2, \ldots$

12. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

    Solution. Putnam Exam, 2001, problem A2 □