Mathematical Induction

The *natural numbers* are the counting numbers: $1, 2, 3, 4, \ldots$. *Mathematical induction* is a technique for proving a statement - a theorem, or a formula - that is asserted about every natural number. For example,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

This asserts that the sum of consecutive numbers from 1 to n is given by the formula on the right. We want to prove that this will be true for $n = 1$, $n = 2$, $n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$:

$$1 + 2 + 3 = \frac{3 \times 4}{2} = 6,$$

which is true. It is also true for $n = 4$:

$$1 + 2 + 3 + 4 = \frac{4 \times 5}{2} = 10.$$

But how are we to prove this rule for *every* value of n? The method of proof is the following:

Principle of Mathematical Induction.
Suppose

1) (The base case) The statement is true for $n = 1$;

2) If the statement is true for n, then it is also true for $n + 1$;

Then the statement is true for every natural number n.

When the statement is true for $n = 1$, then according to 2), it will also be true for $n = 2$. But that implies it will be true for $n = 3$; which implies it will be true for $n = 4$. And so on. It will be true for every natural number. To prove a statement by induction, then, we must prove parts 1) and 2) above.

The hypothesis of part 2) - "The statement is true for n" - is called the inductive assumption, or the inductive hypothesis. It is what we assume when we prove a theorem by induction.

Example 1. Show that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{1}$$

Proof. For $n = 1$, we have $1 = \frac{1(1+1)}{2}$ which is true.

Suppose (the induction hypothesis) that the statement (1) is true for n:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Then

$$\begin{aligned}1+2+3+\ldots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1)\\ &= \frac{n^2+n+2n+2}{2}\\ &= \frac{n^2+3n+2}{2}\\ &= \frac{(n+1)(n+2)}{2},\end{aligned}$$

which proves the statement (1) for $n+1$. By induction, the statement (1) is true for all natural numbers n. □

For the base case of induction, it is not necessary to use $n = 1$. Any other base number k will work, and the result of induction will be that the statement is true for any $n \geq k$.

There is also a technique called *strong induction*, in which the inductive hypothesis is that the statement is true for $1, 2, 3, \ldots, n$.

Problems

1. Prove that $n! > 2^n$ for all $n \geq 4$.

2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

3. Prove that all numbers in the sequence $1007, 10017, 100117, 1001117, 10011117, \ldots$ are divisible by 53.

4. Let F_k be the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$. Show that:

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer.

6. Prove that any square can be dissected into n smaller squares (possibly of differing sizes) for every $n \geq 6$.

7. Show that:

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right),$$

where there are n 2s in the expression on the left.

8. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only.

9. Prove *Bernoulli's Inequality*:

$$(1+x)^n \geq 1+nx$$

for every real number $x \geq -1$ and every natural number n.

10. Prove that $2^{2^n} + 3^{2^n} + 5^{2^n}$ is divisible by 19 for all positive integers n.

11. Prove that $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer for $n = 0, 1, 2, \ldots$

12. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.