Intermediate Value Theorem, Rolle's Theorem and Mean Value Theorem

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In many problems, you are asked to show that something exists, but are not required to give a specific example or formula for the answer. Often in this sort of problem, trying to produce a formula or specific example will be impossible.

The following three theorems are all powerful because they guarantee the existence of certain numbers without giving specific formulas.

Theorem 1 (Intermediate Value Thoerem)**.** *If f is a continuous function on the closed interval $[a, b]$, and if d is between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ with $f(c) = d$.*

As an example, let $f(x) = \cos(x) - x$. Since $f(0) = 1$ and $f(\pi) = -1 - \pi$, there must be a number t between 0 and π with $f(t) = 0$ (so t satisfies $\cos(t) = t$). It is not hard to get a decimal approximation to t but there is no simple formula for t using standard functions.

Theorem 2 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there is a number $c \in [a, b]$ with $f'(c) = 0$.*

Theorem 3 (The Mean Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is a number $c \in [a, b]$ with*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Problems

1. Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x so that

$$f(x) = f\left(x + \frac{1}{n}\right).$$

Solution. Define $g(x) = f(x) - f(x + 1/n)$. Consider the set of numbers $S = \{f(0), f(1/n), f(2/n), \ldots, f(1)\}$. Let k be such that $f(k/n)$ is the largest number in S. Suppose that $k \neq 0$ and $k \neq n$. Then $g(k/n) = f(k/n) - f((k+1)/n) \geq 0$, and $g((k-1)/n) = f((k-1)/n) - f(k/n) \leq 0$. By the IVT, there is $c \in [(k-1)/n, k/n]$ with $g(c) = 0$, so that $f(c) - f(c+1/n) = 0$, or $f(c) = f(c+1/n)$ as desired.

Finally, if the largest number in S is $f(0) = f(1)$, then the same argument works with k chosen so that $f(k/n)$ is the minimum number in S. And note that if $f(0)$ is both the largest and smallest number in S, then they are all the same and $f(0) = f(1/n)$. □

2. Given any two triangles in the plane, show that there is one line that bisects both of them.

   Solution. Call the triangles T_1 and T_2. Let C be the point which is the center of mass of T_1. For $\theta \in [0, 2\pi]$, let ℓ_θ be the oriented line through C whose positive direction makes an angle θ with the horizontal (a picture here would really help). Every ℓ_θ bisects the first triangle.

   Now let $f(\theta)$ be the portion of area of the second triangle which lies to the left of the oriented line ℓ_θ. The function f is continuous, and satisfies $f(\theta)+f(\theta+\pi) = \text{Area}(T_2)$. In particular, $f(0)+f(\pi) = \text{Area}(T_2)$. If $f(0) = \text{Area}(T_2)/2$, we are done, and ℓ_0 bisects both triangles. If $f(0) < \text{Area}(T_2)/2$ then $f(\pi) > \text{Area}(T_2)/2$ and by the intermediate value theorem, there is a θ with $f(\theta) = \text{Area}(T_2)/2$, so that ℓ_θ bisects both triangles. Finally, the case $f(0) > \text{Area}(T_2)/2$ has $f(\pi) < \text{Area}(T_2)/2$ and again by the IVT we get a line bisecting both triangles. □

3. A hiker begins a backpacking trip at 6am on Saturday morning, arriving at camp at 6pm that evening. The next day, the hiker returns on the same trail leaving at 6am in the morning and finishing at 6pm. Show that there is some place on the trail that the hiker visited at the same time of day both coming and going.

   Solution. Let $u(t)$ be the hiker's distance from the trailhead on the uphill trip as a function of time, and $d(t)$ be the hiker's distance from the trailhead on the downhill trip as a function of time. Let $f(t) = d(t) - u(t)$. Now $f(6am) > 0$ and $f(6pm) < 0$, so by the IVT there is a time c with $f(c) = 0$, which means $d(c) = u(c)$, so the hiker ais at the same location on the trail on both trips at time c. □

4. Let a, b, and c be real numbers. Show that the equation

$$4ax^3 + 3bx^2 + 2cx = a + b + c$$

   always has a root between 0 and 1.

Solution. Let $f(x) = ax^4 + bx^3 + cx^2$. Then $f(0) = 0$ and $f(1) = a+b+c$. By the mean value theorem, there is an x_0 between 0 and 1 with $f'(x_0) = a + b + c$, so $4ax_0^3 + 3bx_0^2 + 2cx_0 = a + b + c$ as desired. □

5. Prove that $x^3 - 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.

   Solution. Let $f(x) = x^3 - 3x + c$. If $f(x)$ has roots a and b on $[0, 1]$ then by Rolle's Theorem, there is $c \in (a, b)$ with $f'(c) = 0$. But $f'(x) = 3x^2 - 3$ is not zero for any $c \in (0, 1)$, a contradiction, so f has at most one root in $[0, 1]$. □

6. For n any positive integer and x, y real numbers, find all solutions to $(x^n + y^n) = (x + y)^n$.

   Solution. The only solutions are $y = 0$ or $x = 0$ or, when n is odd, $x = -y$. Fix y, and let $f(x) = x^n + y^n - (x + y)^n$. Compute $f'(x) = n(x^{n-1} - (x + y)^{n-1})$. If $f'(x) = 0$ then $x^{n-1} = (x + y)^{n-1}$ and taking $n - 1$-th roots gives the solution $x = x + y$, which works for any x as long as $y = 0$. When n is odd, there is the additional solution $-x = x + y$ or $x = -y/2$.

   Suppose n is even. We have $f(0) = 0$, and if there is any other x_0 with $f(x_0) = 0$, then by Rolle's theorem, there is some c between 0 and x_0 with $f'(c) = 0$, which can only happen when $y = 0$. We have shown the only solutions are $y = 0$ or $x = 0$ for n even.

   Suppose n is odd. We have $f(0) = 0$ and $f(-y) = 0$. If there is a third solution x_0 with $f(x_0) = 0$ then by Rolle's theorem, there are two distinct solutions for $f'(x) = 0$, which can only happen when $y = 0$. We have shown the only solutions are $x = 0$, $x = -y$, or $y = 0$ for n odd. □

7. Prove that for $0 \leq a < b < \pi/2$,

$$\frac{b-a}{\cos^2(a)} < \tan b - \tan a < \frac{b-a}{\cos^2(b)}.$$

8. Suppose at time $t = 0$, a particle is at rest. At time $t = 1$, the particle is at rest 1 unit from its starting position. Prove that at some moment the particle's accelleration was 4.

9. For which real numbers k does there exist a continuous real valued function f satisfying $f(f(x)) = kx^9$ for all real x?

   Solution. A solution exists if and only if $k \geq 0$. If $k \geq 0$, let $f(x) = k^{1/4}x^3$. Then $f(f(x)) = kx^9$ as desired. Now suppose $k < 0$, and suppose there exists f with $f(x) = kx^9$ for all x.

First, claim $f(0) = 0$. We have $f(f(0)) = k0^9 = 0$, so

$$f(0) = f(f(f(0))) = k(f(0))^9.$$

Then $f(0) = 0$ or $1 = k(f(0))^8$, but the latter is impossible for negative k, so $f(0) = 0$.

Next, we produce a $c \neq 0$ with $f(c) = 0$. Consider $f(1)$, which may be zero, positive, or negative.

- If $f(1) = 0$, let $c = 1$.
- If $f(1) > 0$, then let $a = 1$ and $b = f(1)$. Then $f(a) = f(1) > 0$ and $f(b) = f(f(1)) = k < 0$, so by the IVT there is a c between a and b with $f(c) = 0$. $c \neq 0$ since a and b are both positive.
- If $f(1) < 0$, then let $a = f(1) < 0$ and $b = f(f(1)) = k < 0$. Then $f(a) = f(f(1)) = k < 0$, and $f(b) = f(f(a)) = ka^9 > 0$, so by the IVT, there is c between a and b with $f(c) = 0$. $c \neq 0$ since a and b are both negative.

Finally, we have a contradiction, since $kc^9 = f(f(c)) = f(0) = 0$ implies $c = 0$. This shows f cannot exist for $k < 0$. □

10. Let $f(x)$ be differentiable on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n, show that there exist distinct points $x_1, x_2, \ldots, x_n$ in $[0, 1]$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

11. (MCMC 2004 II.1) Suppose f is a continuous real-valued function on the interval $[0, 1]$. Show that

$$\int_0^1 x^2 f(x)\, dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.

Solution. Because f is continuous, it attains its minimum and maximum at points a and b, both in $[0, 1]$, giving

$$f(a) \int_0^1 x^2 dx \leq \int_0^1 x^2 f(x)\, dx \leq f(b) \int_0^1 x^2\, dx$$

or

$$f(a) \leq 3 \int_0^1 x^2 f(x)\, dx \leq f(b).$$

Thus, the Intermediate Value Theorem guarantees a point $\xi \in [0, 1]$ such that

$$f(\xi) = 3 \int_0^1 x^2 f(x)\, dx.$$

(From Problem 1.5.2, *Berkeley Problems in Mathematics*, Springer, 1998)

□