

Intermediate Value Theorem, Rolle's Theorem and Mean Value Theorem

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In many problems, you are asked to show that something exists, but are not required to give a specific example or formula for the answer. Often in this sort of problem, trying to produce a formula or specific example will be impossible.

The following three theorems are all powerful because they guarantee the existence of certain numbers without giving specific formulas.

Theorem 1 (Intermediate Value Theorem). *If f is a continuous function on the closed interval $[a, b]$, and if d is between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ with $f(c) = d$.*

As an example, let $f(x) = \cos(x) - x$. Since $f(0) = 1$ and $f(\pi) = -1 - \pi$, there must be a number t between 0 and π with $f(t) = 0$ (so t satisfies $\cos(t) = t$). It is not hard to get a decimal approximation to t but there is no simple formula for t using standard functions.

Theorem 2 (Rolle's Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b) , and suppose that $f(a) = f(b)$. Then there is a number $c \in [a, b]$ with $f'(c) = 0$.*

Theorem 3 (The Mean Value Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b) . Then there is a number $c \in [a, b]$ with*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Problems

1. Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x so that

$$f(x) = f\left(x + \frac{1}{n}\right).$$

2. Given any two triangles in the plane, show that there is one line that bisects both of them.

3. A hiker begins a backpacking trip at 6am on Saturday morning, arriving at camp at 6pm that evening. The next day, the hiker returns on the same trail leaving at 6am in the morning and finishing at 6pm. Show that there is some place on the trail that the hiker visited at the same time of day both coming and going.
4. Let a , b , and c be real numbers. Show that the equation

$$4ax^3 + 3bx^2 + 2cx = a + b + c$$

always has a root between 0 and 1.

5. Prove that $x^3 - 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.
6. For n any positive integer and x, y real numbers, find all solutions to $(x^n + y^n) = (x + y)^n$.
7. Prove that for $0 \leq a < b < \pi/2$,

$$\frac{b-a}{\cos^2(a)} < \tan b - \tan a < \frac{b-a}{\cos^2(b)}.$$

8. Suppose at time $t = 0$, a particle is at rest. At time $t = 1$, the particle is at rest 1 unit from its starting position. Prove that at some moment the particle's acceleration was 4.
9. For which real numbers k does there exist a continuous real valued function f satisfying $f(f(x)) = kx^9$ for all real x ?
10. Let $f(x)$ be differentiable on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n , show that there exist distinct points x_1, x_2, \dots, x_n in $[0, 1]$ such that

$$\sum_{i=1}^n \frac{1}{f'(x_i)} = n.$$

11. (MCMC 2004 II.1) Suppose f is a continuous real-valued function on the interval $[0, 1]$. Show that

$$\int_0^1 x^2 f(x) dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.