The Greatest Integer function.

Definition. For a real number x, denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.

A couple of trivial facts about $\lfloor x \rfloor$:

- $\lfloor x \rfloor$ is the unique integer satisfying $x - 1 < \lfloor x \rfloor \leq x$.
- $\lfloor x \rfloor = x$ if and only if x is an integer.
- Any real number x can be written as $x = \lfloor x \rfloor + \theta$, where $0 \leq \theta < 1$.

Some basic properties, with proofs left to the reader:

Proposition 1. *For x a real number and n and integer:*

1. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.
2. $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x = \lfloor x \rfloor, \\ -\lfloor x \rfloor - 1 & \text{if } x \neq \lfloor x \rfloor. \end{cases}$
3. $\lfloor x/n \rfloor = \lfloor \lfloor x \rfloor / n \rfloor$ *if* $n \geq 1$.
4. $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$. *More generally,*

$$\lfloor nx \rfloor = \sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor .$$

The Legendre formula gives the factorization of $n!$ into primes:

Theorem 1 (Legendre Formula)**.** *For n a positive integer,*

$$n! = \prod_{p prime, p \leq n} p^{\alpha(p)}$$

where

$$\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

Note that the sum for $\alpha(p)$ is finite, and that $\alpha(p)$ is the highest power of p that divides $n!$.

Proof. Among the first n positive integers, those divisible by p are $p, 2p, \ldots, tp$, where t is the largest integer such that $tp \leq n$; in other words, t is the largest integer less than or equal to n/p, so $t = \lfloor n/p \rfloor$. Thus there are exactly $\lfloor n/p \rfloor$ multiples of p occurring in the product that defines $n!$, and they are

$$p, 2p, 3p, \ldots, \left\lfloor \frac{n}{p} \right\rfloor p.$$

With the same reasoning, the numbers between 1 and n which are divisible by p^2 are

$$p^2, 2p^2, \ldots, \left\lfloor \frac{n}{p^2} \right\rfloor p^2$$

and there are $\lfloor n/p^2 \rfloor$ of these. Generally, $\lfloor n/p^k \rfloor$ are divisible by p^k and so the total number of times p divides $n!$ is

$$\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

□

All of this material can be found in a good book on number theory, for example Burton, Elementary Number Theory. A deeper treatment is in Apostol, Introduction to Analytic Number Theory.

Exercises

1. Prove the statements in Proposition 1.
2. If $0 < y < 1$, what are the possible values of $\lfloor x \rfloor - \lfloor x - y \rfloor$?

   Solution. Always 1 is possible, and also 0 unless x is an integer. □

3. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. What are the possible values of $\{x\} + \{-x\}$?

   Solution. 0 if x is an integer, 1 otherwise. □

4. Prove that $\lfloor 2x \rfloor - 2 \lfloor x \rfloor$ is either 0 or 1.
5. Prove that $\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$.
6. For an integer $n \geq 0$, prove that $\lfloor n/2 \rfloor - \lfloor -n/2 \rfloor = n$.
7. For an integer $n \geq 1$, the number of digits (in base ten) of n is $1 + \lfloor \log_{10}(n) \rfloor$.

Problems

1. How many zeros does the number 1000! end with?

Solution. One must find out how many factors of 10 are in 1000!. There will be more factors of 2 than 5, so compute the number of factors of five:

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249.$$

There are 249 zeros at the end of 1000!. □

2. If n is a positive integer, prove that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$.

   Solution. First, $(\sqrt{n}+\sqrt{n+1})^2 = 2n+2\sqrt{n^2+n}+1$. Now, $n^2 < n^2+n < (n+1/2)^2$ so that $n < \sqrt{n^2+n} < n+1/2$. Then,

$$\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}.$$

   Squares are always odd or divisible by 4, so $4n+2$ is never a square. Then $\lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$ and so $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$. □

3. Determine all positive integers n such that $\lfloor \sqrt{n} \rfloor$ divides n.

   Solution. When $n = k^2$, $n = k(k+1)$, or $n = k(k+2) = (k+1)^2 - 1$. □

4. If n is a positive integer, prove that

$$\left\lfloor \frac{8n+13}{25} \right\rfloor - \left\lfloor \frac{n - 12 - \left\lfloor \frac{n-17}{25} \right\rfloor}{3} \right\rfloor$$

   is independent of n.

   Solution. Bring the first term inside, get a common denominator. Check that it's periodic mod 25, then check for $n = 0, \ldots, 24$. □

5. Prove that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

   Solution. Easy to do in two cases, n even or n odd, using $1 + \cdots + m = \frac{m(m+1)}{2}$. □

6. A sequence of real numbers is defined by the nonlinear first order recurrence

$$u_{n+1} = u_n(u_n^2 - 3).$$

   (a) If $u_0 = 5/2$, give a simple formula for u_n.
   (b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?

Solution. Prove by induction that if $u_0 = \frac{d^2+1}{d}$ then $u_n = \frac{d^{2\cdot 3^n}+1}{d^{3^n}}$. In particular, for $u_0 = 5/2$, $u_n = \frac{4^{3^n}+1}{2^{3^n}}$. For part (b), there are $\lfloor 3^{10}\log_{10}(2+\sqrt{3})\rfloor + 1$ digits. (MIT 18.S34 F'07). □

7. Which positive integers can be written in the form $n + \lfloor \sqrt{n} + 1/2 \rfloor$ for some positive integer n?

   Solution. Looks like all but the squares. Haven't proved it yet. (MIT 18.S34 F'07). □

8. Prove that the sequence $\left\{\lfloor (\sqrt{2})^n \rfloor\right\}_{n=0}^{\infty}$ contains infinitely many odd numbers.

   Solution. $(\sqrt{2})^n$ is even (a power of 2) when n is even. When $n = 2k+1$ is odd, $(\sqrt{2})^n = 2^k\sqrt{2}$. Now $\lfloor 2^k\sqrt{2}\rfloor$ is odd exactly when the k^{th} binary digit of $\sqrt{2}$ is 1. Since $\sqrt{2}$ is irrational, its binary expansion must have infinitely many 1's. (Original, inspired by the Graham-Pollak sequence). □

9. Determine whether the improper integral

$$\int_0^\infty (-1)^{\lfloor x^2 \rfloor} dx$$

   converges or diverges, where $\lfloor \cdot \rfloor$ is the greatest integer function.

   Solution. It converges. For $\sqrt{n} \le x < \sqrt{n+1}$, we have $\lfloor x^2 \rfloor = n$, so that

$$\int_0^\infty (-1)^{\lfloor x^2 \rfloor} dx = \sum_{n=0}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}).$$

   Since the series is alternating and its terms approach zero, it converges. Source: Youngstown State Calculus Competition, 2005. □

10. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

   (Here $\min(a, b)$ denotes the minimum of a and b.)

   Solution. Putnam 1997 B1 □

11. Let a, b, c, d be real numbers such that $\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor$ for all positive integers n. Prove that at least one of $a + b$, $a - c$, $a - d$ is an integer.

Solution. (MIT 18.S34 F'07) □

12. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \le i \le n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is not allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$$

Find a simple formula for a_n. Your formula should enable you, for instance, to compute $a_{1000000}$.

Solution. (MIT 18.S34 F'07) □

13. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

Solution. Very hard! (Putnam 1995 B6) □