The Greatest Integer function.

Definition. For a real number x, denote by $\lfloor x \rfloor$ the largest integer less than or equal to x.

A couple of trivial facts about $\lfloor x \rfloor$:

- $\lfloor x \rfloor$ is the unique integer satisfying $x - 1 < \lfloor x \rfloor \leq x$.
- $\lfloor x \rfloor = x$ if and only if x is an integer.
- Any real number x can be written as $x = \lfloor x \rfloor + \theta$, where $0 \leq \theta < 1$.

Some basic properties, with proofs left to the reader:

Proposition 1. *For x a real number and n and integer:*

1. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.
2. $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x = \lfloor x \rfloor, \\ -\lfloor x \rfloor - 1 & \text{if } x \neq \lfloor x \rfloor. \end{cases}$
3. $\lfloor x/n \rfloor = \lfloor \lfloor x \rfloor / n \rfloor$ *if* $n \geq 1$.
4. $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$. *More generally,*

$$\lfloor nx \rfloor = \sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor .$$

The Legendre formula gives the factorization of $n!$ into primes:

Theorem 1 (Legendre Formula)**.** *For n a positive integer,*

$$n! = \prod_{p prime, p \leq n} p^{\alpha(p)}$$

where

$$\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

Note that the sum for $\alpha(p)$ is finite, and that $\alpha(p)$ is the highest power of p that divides $n!$.

Proof. Among the first n positive integers, those divisible by p are $p, 2p, \ldots, tp$, where t is the largest integer such that $tp \leq n$; in other words, t is the largest integer less than or equal to n/p, so $t = \lfloor n/p \rfloor$. Thus there are exactly $\lfloor n/p \rfloor$ multiples of p occurring in the product that defines $n!$, and they are

$$p, 2p, 3p, \ldots, \left\lfloor \frac{n}{p} \right\rfloor p.$$

With the same reasoning, the numbers between 1 and n which are divisible by p^2 are

$$p^2, 2p^2, \ldots, \left\lfloor \frac{n}{p^2} \right\rfloor p^2$$

and there are $\lfloor n/p^2 \rfloor$ of these. Generally, $\lfloor n/p^k \rfloor$ are divisible by p^k and so the total number of times p divides $n!$ is

$$\alpha(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

□

All of this material can be found in a good book on number theory, for example Burton, Elementary Number Theory. A deeper treatment is in Apostol, Introduction to Analytic Number Theory.

Exercises

1. Prove the statements in Proposition 1.
2. If $0 < y < 1$, what are the possible values of $\lfloor x \rfloor - \lfloor x - y \rfloor$?
3. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. What are the possible values of $\{x\} + \{-x\}$?
4. Prove that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is either 0 or 1.
5. Prove that $\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$.
6. For an integer $n \geq 0$, prove that $\lfloor n/2 \rfloor - \lfloor -n/2 \rfloor = n$.
7. For an integer $n \geq 1$, the number of digits (in base ten) of n is $1 + \lfloor \log_{10}(n) \rfloor$.

Problems

1. How many zeros does the number 1000! end with?
2. If n is a positive integer, prove that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$.
3. Determine all positive integers n such that $\lfloor \sqrt{n} \rfloor$ divides n.
4. If n is a positive integer, prove that

$$\left\lfloor \frac{8n+13}{25} \right\rfloor - \left\lfloor \frac{n - 12 - \left\lfloor \frac{n-17}{25} \right\rfloor}{3} \right\rfloor$$

   is independent of n.

5. Prove that
$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor .$$

6. A sequence of real numbers is defined by the nonlinear first order recurrence
$$u_{n+1} = u_n(u_n^2 - 3).$$
(a) If $u_0 = 5/2$, give a simple formula for u_n.
(b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?

7. Which positive integers can be written in the form $n + \lfloor \sqrt{n} + 1/2 \rfloor$ for some positive integer n?

8. Prove that the sequence $\left\{ \lfloor (\sqrt{2})^n \rfloor \right\}_{n=0}^{\infty}$ contains infinitely many odd numbers.

9. Determine whether the improper integral
$$\int_0^{\infty} (-1)^{\lfloor x^2 \rfloor} dx$$
converges or diverges, where $\lfloor \cdot \rfloor$ is the greatest integer function.

10. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate
$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right).$$
(Here $\min(a, b)$ denotes the minimum of a and b.)

11. Let a, b, c, d be real numbers such that $\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor$ for all positive integers n. Prove that at least one of $a + b$, $a - c$, $a - d$ is an integer.

12. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \le i \le n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is not allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:
$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$$
Find a simple formula for a_n. Your formula should enable you, for instance, to compute $a_{1000000}$.

13. For a positive real number α, define
$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}.$$
Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.