The Fundamental Theorem of Calculus.

The two main concepts of calculus are integration and differentiation. The Fundamental Theorem of Calculus (FTC) says that these two concepts are essentially inverse to one another.

The fundamental theorem states that if F has a continuous derivative on an interval $[a, b]$, then

$$\int_a^b F'(t)dt = F(b) - F(a).$$

This form allows one to compute integrals by finding anti-derivatives.

The FTC says that integration undoes differentiation (up to a constant which is irrevocably lost when taking derivatives), in the sense that

$$F(x) = \int_a^x \frac{d}{dt}(F(t))dt + C$$

where $C = F(a)$.

The second part of the fundamental theorem says that differentiation undoes integration, in the sense that

$$f(x) = \frac{d}{dx}\int_a^x f(t)dt,$$

where f is a continuous function on an open interval containing a and x.

Problems

1. Let $f(x) = \frac{1}{1+x^4} + a$, and let F be an antiderivative of f, so that $F' = f$. Find a so that F has exactly one critical point.

   Solution. $a = -1$. Clemson Calculus Competition. $\square$

2. Let

   $$f(x) = \int_x^2 \frac{1}{\sqrt{1+t^3}}dt.$$

   Find

   $$\int_0^2 xf(x)dx.$$

   Solution. Integrate by parts to get

   $$\int_0^2 xf(x)dx = -\int_0^2 \frac{1}{2}x^2 f'(x)dx = \int_0^2 \frac{x^2}{2\sqrt{1+x^3}}dx.$$

   The rest is a straightforward integral after substituting for x^3. The solution is $\frac{2}{3}$. $\square$

3. What function is defined by the equation

$$f(x) = \int_0^x f(t)dt + 1?$$

Solution. Larson 6.9.7 $\square$

4. Let f be such that

$$x \sin(\pi x) = \int_0^{x^2} f(t)dt.$$

Find $f(4)$.

Solution. Put $F(x) = \int_0^x f(t)dt$. Then $F(x^2) = x\sin(\pi x)$. Take the derivative of both sides to get $2xf(x) = \sin(\pi x) + \pi x\cos(\pi x)$ and plug in $x = 2$ to find $f(4) = \pi/2$. Kenneth Roblee, Troy U. $\square$

5. If a, b, c, d are polynomials, show that

$$\int_1^x a(x)c(x)dx \int_1^x b(x)d(x)dx - \int_1^x a(x)d(x)dx \int_1^x b(x)c(x)dx$$

is divisible by $(x-1)^4$.

Solution. Denote the expression in question by $F(x)$. F is a polynomial. We know $(x-1)^4$ divides F if and only if $F'''(1) = 0$. Check this by differentiation. (Larson 6.9.3). $\square$

6. Suppose that f is differentiable, and that $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, prove that $f(x)/x$ is strictly increasing for $x > 0$.

Solution. Larson 6.9.12 $\square$

7. (MCMC 2005 II.5) Suppose that $f\colon [0,\infty) \to [0,\infty)$ is a differentiable function with the property that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal to the arclength of the curve $y = f(x)$ from $x = a$ to $x = b$. Given that $f(0) = 5/4$, and that $f(x)$ has a minimum value on the interval $(0,\infty)$, find that minimum value.

Solution. The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(t)\,dt,$$

and the arclength of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1 + (f'(t))^2}\,dt.$$

Therefore,
$$\int_a^b f(t)\,dt = \int_a^b \sqrt{1+(f'(t))^2}\,dt$$
for all nonnegative a and b. In particular, we can write
$$\int_0^x f(t)\,dt = \int_0^x \sqrt{1+(f'(t))^2}\,dt$$
for all nonnegative x. Both sides of the above equation define a function of x, and since they are equal, their derivatives are equal; their derivatives are given by the Second Fundamental Theorem of Calculus:
$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\left(\int_0^x \sqrt{1+(f'(t))^2}\,dt\right),$$
i.e.,
$$f(x) = \sqrt{1+(f'(x))^2}.$$
So, we are looking for a function y which satisifies the differential equation
$$y = \sqrt{1+(y')^2}.$$
This equation is separable:
$$\begin{aligned} y = \sqrt{1+(y')^2} &\Rightarrow y^2 = 1+(y')^2 && (1)\\ &\Rightarrow (y')^2 = y^2-1 && (2)\\ &\Rightarrow y' = \sqrt{y^2-1} && (3)\\ &\Rightarrow \frac{dy}{\sqrt{y^2-1}} = dx. && (4) \end{aligned}$$
Integrating both sides yields
$$\int \frac{dy}{\sqrt{y^2-1}} = \int dx \Rightarrow \ln\left|y+\sqrt{y^2-1}\right| = x + C$$
(where the first integral is evaluated using the trig substitution $y = \sec\theta$ and the two arbitrary constants of integration are combined into one constant on the right hand side). Next, since $f(0) = 5/4$ is positive, we can drop the absolute value, and solve for y:
$$\begin{aligned} \ln\big(y+\sqrt{y^2-1}\big) = x + C \Rightarrow y + \sqrt{y^2-1} = e^{x+C} = Ae^x \quad (\text{where } A = e^C) \\ \Rightarrow \sqrt{y^2-1} = Ae^x - y && (5)\\ \Rightarrow y^2 - 1 = (Ae^x - y)^2 = A^2e^{2x} - 2Aye^x + y^2 && (6)\\ \Rightarrow -1 = A^2e^{2x} - 2Aye^x && (7)\\ \Rightarrow 2Aye^x = A^2e^{2x} + 1 && (8)\\ \Rightarrow y = \frac{A^2e^{2x}+1}{2Ae^x} = \frac{A}{2}e^x + \frac{1}{2A}e^{-x}. && (9) \end{aligned}$$

Using $f(0) = 5/4$, we find

$$\frac{5}{4} = \frac{A}{2} + \frac{1}{2A} \Rightarrow A = \frac{1}{2} \quad \text{or} \quad 2.$$

This gives two possible functions:

$$y = \frac{1}{4}e^x + e^{-x} \quad \text{or} \quad y = e^x + \frac{1}{4}e^{-x}.$$

This latter has a minimum at $x = -\ln 2$, which is not positive, so we reject that function. The former has a minimum at $x = \ln 2$, and the y value is 1. Note: One could also deduce from the differential equation $y' = \sqrt{y^2 - 1}$ that at the minimum value, since $y' = 0$, the y-value must be 1. □

8. (MCMC 2006 I.5) Let $f(t)$ and $f'(t)$ be differentiable on $[a, x]$ and for each x suppose there is a number c_x such that $a < c_x < x$ and

$$\int_a^x f(t)\,dt = f(c_x)(x - a).$$

Assume that $f'(a) \neq 0$. Then prove that

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

Solution. Let

$$F(x) = \int_a^x f(t)\,dt.$$

Using Taylor's expansion of $F(x)$, we have

$$F(x) = F(a) + (x - a)F'(a) + \frac{(x - a)^2}{2}F''(\theta_x),$$

where θ_x lies strictly between a and x, and as x goes to a, θ_x also goes to a. We also have $F(a) = 0$, $F'(x) = f(x)$, and $F''(x) = f'(x)$. Thus,

$$F(x) = 0 + (x - a)f(a) + \frac{(x - a)^2}{2}f'(\theta_x).$$

By definition,

$$f(c_x) = \frac{1}{x - a}F(x) = f(a) + \frac{x - a}{2}f'(\theta_x).$$

Therefore,

$$\frac{f(c_x) - f(a)}{x - a} = \frac{1}{2}f'(\theta_x).$$

On the other hand we can write

$$\frac{f(c_x) - f(a)}{x - a}$$

as a product

$$\frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x - a}.$$

On taking the limits of these as x goes to a, we get

$$\lim_{x \to a} \frac{1}{2} f'(\theta_x) = \lim_{x \to a} \frac{f(c_x) - f(a)}{c_x - a} \cdot \frac{c_x - a}{x - a}.$$

This gives

$$\frac{1}{2} f'(a) = \lim_{x \to a} \frac{f(c_x) - f(a)}{c_x - a} \lim_{x \to a} \frac{c_x - a}{x - a}.$$

In other words,

$$\frac{1}{2} f'(a) = f'(a) \cdot \lim_{x \to a} \frac{c_x - a}{x - a}.$$

This shows

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

$\square$