The Fundamental Theorem of Calculus.

The two main concepts of calculus are integration and differentiation. The Fundamental Theorem of Calculus (FTC) says that these two concepts are essentially inverse to one another.

The fundamental theorem states that if F has a continuous derivative on an interval $[a, b]$, then

$$\int_a^b F'(t)dt = F(b) - F(a).$$

This form allows one to compute integrals by finding anti-derivatives.

The FTC says that integration undoes differentiation (up to a constant which is irrevocably lost when taking derivatives), in the sense that

$$F(x) = \int_a^x \frac{d}{dt}(F(t))dt + C$$

where $C = F(a)$.

The second part of the fundamental theorem says that differentiation undoes integration, in the sense that

$$f(x) = \frac{d}{dx}\int_a^x f(t)dt,$$

where f is a continuous function on an open interval containing a and x.

Problems

1. Let $f(x) = \frac{1}{1+x^4} + a$, and let F be an antiderivative of f, so that $F' = f$. Find a so that F has exactly one critical point.

2. Let

   $$f(x) = \int_x^2 \frac{1}{\sqrt{1+t^3}}dt.$$

   Find

   $$\int_0^2 xf(x)dx.$$

3. What function is defined by the equation

   $$f(x) = \int_0^x f(t)dt + 1?$$

4. Let f be such that

   $$x\sin(\pi x) = \int_0^{x^2} f(t)dt.$$

   Find $f(4)$.

5. If a, b, c, d are polynomials, show that

$$\int_1^x a(x)c(x)dx \int_1^x b(x)d(x)dx - \int_1^x a(x)d(x)dx \int_1^x b(x)c(x)dx$$

is divisible by $(x-1)^4$.

6. Suppose that f is differentiable, and that $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, prove that $f(x)/x$ is strictly increasing for $x > 0$.

7. (MCMC 2005 II.5) Suppose that $f\colon [0,\infty) \to [0,\infty)$ is a differentiable function with the property that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal to the arclength of the curve $y = f(x)$ from $x = a$ to $x = b$. Given that $f(0) = 5/4$, and that $f(x)$ has a minimum value on the interval $(0,\infty)$, find that minimum value.

8. (MCMC 2006 I.5) Let $f(t)$ and $f'(t)$ be differentiable on $[a, x]$ and for each x suppose there is a number c_x such that $a < c_x < x$ and

$$\int_a^x f(t)\,dt = f(c_x)(x-a).$$

Assume that $f'(a) \neq 0$. Then prove that

$$\lim_{x\to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$