11. (*On the complexity of Euler's constant*) *Euler's constant* or the *Euler–Mascheroni constant* γ is defined by

$$\gamma := \lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\log m\right]=0.5772156649\ldots$$

(10.2.20)

It is related to the gamma function by the formula

(10.2.21) $$\frac{1}{\Gamma(z)}=ze^{\gamma z}\left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right].$$

a) Use the recursion

$$\Gamma(z+1)=z\Gamma(z)$$

to prove that if Γ has an expansion of form (10.2.21), then γ is given by (10.2.20).

b) The *exponential integral* E_1 is defined by

(10.2.22) $$E_1(x)=\int_x^{\infty}\frac{e^{-t}\,dt}{t}.$$

Show that

(10.2.23) $$-E_1(x)=\gamma+\log x+\sum_{k=1}^{\infty}\frac{(-x)^k}{k\cdot k!}\qquad x>0.$$

c) Use (10.2.23) to construct an

$$O_B((\log n)^2M(n))$$

algorithm for γ by choosing x roughly of size $6n$.

This method, suggested by Sweeney [63], is a reasonably efficient method for computing γ. Brent and McMillan [80] present a number of algorithms for this computation. They calculate over 29,000 partial quotients of the continued fraction for γ. As a consequence they show that if γ is rational the denominator of γ exceeds $10^{15,000}$.

Chapter Eleven

Pi

Abstract. *The first section of the chapter deals with the history of the calculation of π and related matters, while the second section deals with its transcendence. The third section looks at irrationality measures and includes a proof of the irrationality of $\zeta(3)$. This chapter is largely self-contained and indeed contains considerable related number theory, especially in the exercises.*

11.1 *ON THE HISTORY OF THE CALCULATION OF π*

The history of π presumably begins with man's first attempts at estimating the perimeter or area of a circle of given radius and as such starts at the dawn of recorded history. The Egyptian Rhind (or Ahmes) Papyrus which dates from approximately 2000 B.C., gives a value of $(16/9)^2 = 3.1604\ldots$ for π. Various other early Babylonian and Egyptian estimates include 3, $3\frac{1}{8}$, and $3\frac{1}{7}$. Implicit in the Bible (1 Kings 7: 23) is a value 3: "And he made a molten sea, ten cubits from the one brim to the other; it was round all about. . . and a line of thirty cubits did compass it round about."

Mathematical interest in π comes into sharp focus in the classical Greek period. The Greeks investigated the problem of "squaring the circle." This question and its final resolution over two millenia later will be pursued in the next section. Currently we wish to review the primary Western developments in the calculation of π.

Archimedes of Syracuse (287–212 B.C.) provided the first major landmark in the quest for digits of π. By considering inscribed and circumscribed polygons of 96 sides, Archimedes gave the estimate

$$3\frac{10}{71}<\pi<3\frac{1}{7}.$$

A salient feature of Archimedes' method is that it can, in principle, be used to provide any number of digits of π.

If a_n denotes the length of a circumscribed regular $6\cdot 2^n$-gon and b_n denotes the length of an inscribed regular $6\cdot 2^n$-gon about a circle of radius 1/2, then

(11.1.1) $$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}$$

(11.1.2) $$b_{n+1} = \sqrt{a_{n+1} b_n}\,.$$

This two-term iteration, starting with $a_0 := 2\sqrt{3}$ and $b_0 := 3$, can be used to calculate π. (See also Section 8.4.) The fourth iteration yields $a_4 = 3.1427\ldots$ and $b_4 = 3.1410\ldots$ and corresponds to estimating π using polygons with 96 sides.

If we observe that

(11.1.3) $$a_{n+1} - b_{n+1} = \frac{a_{n+1} b_n}{(a_{n+1} + b_{n+1})(a_n + b_n)}(a_n - b_n)$$

we again see that the error is decreased by a factor of approximately 4 with each iteration. Variations of this modern formulation of Archimedes' method provided the basis for virtually all extended precision calculations of π for the next 1800 years, culminating with Ludolph van Ceulen (1540–1610) who correctly computed 34 digits. The limitations of this method stem from the relatively slow convergence and from the need to extract square roots. (See Exercise 1.)

François Vièta (1540–1603) gave the first infinite expansion

(11.1.4) $$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

which he derived by considering a limit of areas of inscribed 2^n-gons. (See Exercise 2.) John Wallis (1616–1703) through a complicated calculation demanding prodigious numerical insight derived the infinite product expansion

(11.1.5) $$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}\,.$$

This appears in his *Arithmetica Infinitorum* of 1655. A few years later Lord Brouncker (1620–1684), the first president of the Royal Society, recast this as the continued fraction.

(11.1.6) $$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}\,.$$

25 missing

The Scottish mathematician James Gregory (1638–1675) in 1671 provided the underlying method for the next era in the history of the calculation of π. He showed that

(11.1.7) $$\arctan x = \int_0^x \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

and hence, on setting $x := 1$, that

(11.1.8) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

a formula independently discovered in 1674 by Leibniz (1646–1716). By the beginning of the eighteenth century Abraham Sharp under the direction of the English astronomer and mathematician E. Halley had obtained 71 correct digits of π using Gregory's series (11.1.7) with $x := \sqrt{1/3}$, namely,

(11.1.9) $$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{3^2\cdot 5} - \frac{1}{3^3\cdot 7} + \cdots\right).$$

It is the techniques of calculus that so expanded the scope for calculating, and it is perhaps not surprising that Isaac Newton (1642–1727) himself calculated π to 15 digits sometime in 1665–66. He used the series

(11.1.10) $$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

which is essentially an arcsin expansion. (See Exercise 4.) Newton was later to write: "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time." John Machin (1680–1752) derived the formula which bears his name:

(11.1.11) $$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

Coupled with Gregory's series for arctan this provides a very attractive method for calculating π since the first term is well suited to decimal arithmetic and the second term converges very rapidly. Machin calculated 100 digits this way in 1706. In the same year William Jones published his *A*

New Introduction to the Mathematics, where he denoted the ratio of the circumference to the diameter by the Greek letter π, presumably for the first letter of periphery. It was, however, Leonard Euler (1707–1783) who popularized the use of the symbol. Euler derived numerous series and products for π and π^2. Among the best-known are

(11.1.12) $$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and

(11.1.13) $$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

The explicit summation of (11.1.12) had eluded Liebniz and also the Bernoulli brothers, Jacques and Jean. The method by which Euler derived his evaluations of $\Sigma_{n=1}^{\infty} 1/n^{2k}$ is outlined in Exercise 7. This is to be found in Euler's *Introducio in Analysin Infinitorum* of 1748. The Machin-like formula

(11.1.14) $$\pi = 20 \arctan\left(\frac{1}{7}\right) + 8 \arctan\left(\frac{3}{79}\right)$$

coupled with the expansion

(11.1.15) $$\arctan x = \frac{y}{x}\left(1 + \frac{2}{3} y + \frac{2 \cdot 4}{3 \cdot 5} y^2 + \cdots\right)$$

where $y := x^2/(1 + x^2)$, allowed Euler to compute 20 digits of π in under an hour.

The next 200 years saw little change in the methods employed to calculate π. In 1844 Johann Dase (1824–1861), a calculating prodogy, used the formula

(11.1.16) $$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$$

to produce 205 digits of π. (Dase's arithmetical abilities were awesome—he could multiply 100-digit numbers together in his head, a feat which took him roughly 8 hours.)

The zenith (or nadir depending on your perspective) in premachine calculations was achieved by William Shanks (1812–1882), who published 607 purported digits of π, of which 527 were correct. Later Shanks published an extension to 707 digits. This was also incorrect after the 527th digit. These calculations took Shanks years and were performed in an entirely straightforward fashion using no tricks or shortcuts. (See W. Shanks [1853].) The mistakes went unnoticed until 1945, when D. F. Ferguson, in one of the final hand calculations, produced 530 digits. Ferguson produced 808 digits in 1947, using a desk calculator and the formula

(11.1.17) $$\frac{\pi}{4} = 3 \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right).$$

Thus dawns the computer age. In June 1949 ENIAC (Electronic Numerical Integrator and Computer) was used to evaluate 2037 digits of π using Machin's formula and 70 hours elapsed time. An analysis of the distribution of the digits was carried out by Metropolis, Reitwiesner, and von Neumann. By 1958, Genuys had computed 10,000 digits on an IBM 704 in 100 minutes, once again using Machin's formula. Felton had performed a 10,000-digit calculation in 1957; however, due to machine error it was only correct to 7480 digits. In 1961 D. Shanks and Wrench [62] used the identity

(11.1.18) $$\pi = 24 \arctan\left(\frac{1}{8}\right) + 8 \arctan\left(\frac{1}{57}\right) + 4 \arctan\left(\frac{1}{239}\right)$$

and under 9 hours on an IBM 7090 to produce 100,000 digits of π. This was checked using the formula

(11.1.19) $$\pi = 48 \arctan\left(\frac{1}{18}\right) + 32 \arctan\left(\frac{1}{57}\right) - 20 \arctan\left(\frac{1}{239}\right).$$

The million-digit mark was set by Guilloud and Bouyer in 1973 on a CDC 7600. The calculation, which took just under a day, used (11.1.19) with (11.1.18) as a check.

Kanada, Tamura, Yoshino, and Ushiro [Pr] calculated in excess of 16 million digits using an AGM based algorithm, Algorithm 2.2, and checked 10 million digits using (11.1.19). The 16 million-digit calculation took under 30 hours on a HITAC M-280H and used an FFT-based fast multiplication.

At the end of 1985 the record belonged to W. Gosper. He calculated 17 million terms of the continued fraction expansion for π and so in excess of this number of decimal digits—after a radix conversion from a binary computation. His method is based on a very careful evaluation of Ramanujan's series (5.5.23) on a Symbolics 3670. (A remarkable feat considering the size of the machine.) As is surprisingly often the case with these large scale calculations, Gosper uncovered subtle design flaws which had not surfaced in smaller calculations.

In January 1986, D. H. Bailey [Pr] computed 29,360,000 decimal digits of π on the CRAY-2 at the NASA Ames Research Center. This calculation used only 12 steps of the quartic algorithm (5.4.7) with $r := 4$. This results in computing $\alpha(2^{50})$, which agrees with π^{-1} to more than 45 million places. The calculation took less than 28 hours and was verified with a 40-hour computation of 25 steps of Algorithm 2.1. It is amusing to observe that the quartic calculation requires well under 100 full precision multiplications, divisions, and root extractions.

In July 1986, Kanada reclaimed the record with a computation of 2^{25} decimal digits. He again used Algorithm 2.2, verified in September using (5.4.7) with $r := 4$, but reduced the elapsed time to 5 hours and 56 minutes on a S-810/20 super computer. This represented a speed-up by a factor of

15. His previous computation now used only 96 minutes of CPU time. Plans were to compute 2^{27} decimal digits (over 100 million) at the end of 1987.

Nor is the end in sight. It will probably be the case than hundreds or thousands of millions of digits will be calculated by the end of the century. (This is now more a matter of will than anything else.) Apart from observations like "the sequence 314159 appears in the digits of π commencing at digit 9,973,760," there is little we care to say about the digits. They have, however, been subjected to considerable scrutiny. It is an open question as to whether π is *normal*. That is, do all sequences of integers appear with the same frequency in the digits (are one-tenth of the digits 7, one-hundredth of the consecutive digit pairs 23, etc.)? On the basis of the first 30 million digits the answer appears positive. This, of course, is no great help in deciding the normality issue. (See Wagon [85].)

In terms of utility, even far-fetched applications such as measuring the circumference of the universe require no more digits than Ludolph van Ceulen had available—but then utility has had little to do with this particular story.

Comments and Exercises

This section presents only the highlights of the quest for digits. The matter may be pursued in detail in Beckmann [77], a most useful though rather individualistic history, and in *Le Petit Archimède* [80]. Schepler's chronography [50] and Wrench's history [60] are also of interest. Details of the more recent calculations may be found in Tamura and Kanada [Pr], where a compendium of Machin-like identities is provided.

There is also a considerable collection of π-related trivia. For example, the Indiana House of Representatives attempted to legislate the value of π in Bill 246 of 1897. The bill, which appears to proclaim π to be several different incredibly inaccurate values, including 4 and 64/25 (see Beckmann [77], and Singmaster [85]), passed the House and only floundered in the Senate on the apparently chance intercession of C. A. Waldo, a professor at Purdue. Keith [86] gives a 402 digit mnemonic for π.

1. a) Show that the algorithm of (11.1.1) and (11.1.2) calculates π by showing that a_n and b_n are as advertised.

b) Prove (11.1.3) and estimate how many iterations of (11.1.1) and (11.1.2) are required to calculate 35 digits of π. This should be compared to Bailey's [Pr] calculation which uses the same operations.

2. a) Prove, from the product expansions for sin and cos, that

$$\theta = \frac{\sin\theta}{\cos(\theta/2)\cos(\theta/2^2)\cos(\theta/2^3)\cdots} \qquad |\theta| < \pi . \tag{11.1.20}$$

b) Alternatively deduce (11.1.20) in an elementary fashion by setting

$$I_n := \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2^2}\right)\cdots\cos\left(\frac{\theta}{2^n}\right)$$

and showing that

$$I_n = \frac{\sin\theta}{2^n \sin(\theta/2^n)} .$$

c) Set $\theta := \pi/2$ and use the formula $\cos(\theta/2) = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta}$ to deduce Vièta's formula (11.1.4)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots .$$

3. a) Prove Wallis's formula (11.1.5) in the form

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots .$$

Hint: Show that

$$\int_0^{\pi/2} \sin^{2m} x\, dx = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\,\frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \sin^{2m+1} x\, dx = \frac{2\cdot 4\cdot 6\cdots 2m}{1\cdot 3\cdot 5\cdots(2m+1)} .$$

b) Establish the corresponding formula for e:

$$\frac{e}{2} = \left(\frac{2}{1}\right)^{1/2}\left(\frac{2\cdot 4}{3\cdot 3}\right)^{1/4}\left(\frac{4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7}\right)^{1/8}\cdots .$$

c) Show that the volume of the $2n$-dimensional unit sphere is $\pi^n/n!$ while the $(2n+1)$-dimensional unit sphere has the volume $2^{2n+1}[n!/(2n+1)!]\pi^n$. Find a unified formula for these two cases.

4. Deduce (11.1.10) roughly as Newton did. Show that

$$\frac{\pi}{24} - \frac{\sqrt{3}}{32} = \int_0^{1/4} \sqrt{x - x^2}\, dx$$

and that

$$\int_0^{1/4} \sqrt{x-x^2}\, dx = \int_0^{1/4} \sqrt{x}(\sqrt{1-x})\, dx$$
$$= \frac{2}{3\cdot 2^3} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \cdots .$$

5. a) Deduce Machin's formula

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

as follows. Let $\theta := \arctan \frac{1}{5}$. Then

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{5}{12}$$

and

$$\tan 4\theta = \frac{120}{119} = 1 + \frac{1}{119}.$$

Hence

$$\tan\left(4\theta - \frac{\pi}{4}\right) = \frac{-1 + \tan 4\theta}{1 + \tan 4\theta} = \frac{1}{239}.$$

b) Show that arctan satisfies the addition formula

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \qquad xy < 1.$$

c) Show that

$$\text{(11.1.21)} \quad \arctan\left(\frac{1}{p}\right) = \arctan\left(\frac{1}{p+q}\right) + \arctan\left(\frac{q}{p^2 + pq + 1}\right)$$

and that if $1 + p^2 = qr$,

$$\text{(11.1.22)} \quad \arctan\left(\frac{1}{p+r}\right) + \arctan\left(\frac{1}{p+q}\right) = \arctan\left(\frac{1}{p}\right).$$

Formula (11.1.21) was known to Euler. Bromwich [26] attributes (11.1.22) to Charles Dodgson (Lewis Carroll).

6. (*Machin-like formulae*)

a) Show, for integral a_j and b_j, that

$$k\pi = \arctan\left(\frac{b_1}{a_1}\right) + \arctan\left(\frac{b_2}{a_2}\right) + \cdots + \arctan\left(\frac{b_n}{a_n}\right)$$

where k is an integer if and only if

$$(a_1 + ib_1)(a_2 + ib_2) \cdots (a_n + ib_n)$$

has zero imaginary part.

Hint: Consider $(a_1 + ib_1) \cdots (a_n + ib_n) = re^{i\theta}$, $|\theta| < \pi$, and use the fact that

$$\arctan z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

(This gives an algorithmic check for Machin-like formulae.)

b) Show, for positive integral u, v, and k and integral m and n, that

$$m \arctan\left(\frac{1}{u}\right) + n \arctan\left(\frac{1}{v}\right) = \frac{k\pi}{4}$$

if and only if $(1-i)^k(u+i)^m(v+i)^n$ is real.

c) Verify

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \qquad \text{(Machin, 1706)}$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) \qquad \text{(Euler, 1738)}$$

$$\frac{\pi}{4} = 2 \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{7}\right) \qquad \text{(Hermann, 1706)}$$

$$\frac{\pi}{4} = 2 \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) \qquad \text{(Hutton, 1776)}.$$

These are, in fact, all the nontrivial solutions of b). This was a problem of Gravé's solved by Størmer in 1897. The problem can be reduced to finding integral solutions of $1 + x^2 = 2y^n$ or $1 + x^2 = y^n$, $n \geq 3$, n odd. (See Ribenboim [84].) Much related material on Machin-like formulae occurs in Lehmer [38] and Todd [49].

7. Prove Brouncker's continued fraction by showing that

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdots}}}.$$

Hint: If

$$s := a_0 + a_1 + a_1a_2 + a_1a_2a_3 + \cdots$$

then

$$s = a_0 + \cfrac{a_1}{1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{1 + a_3 - \cdots}}}.$$

This is a nonsimple continued fraction. The convergents satisfy a similar recursion to that given in Exercise 2 of Section 11.3.

Apply this to

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

so that

$$a_0 = 0,\ a_1 = x,\ a_2 = \frac{-x^2}{3},\ a_3 = \frac{-3x^2}{5}, \ldots$$

and

$$\arctan x = \cfrac{x}{1 + \cfrac{x^2}{3 - x^2 + \cfrac{9x^2}{5 - 3x^2 + \cfrac{25x^2}{7 - 5x^2 \cdots}}}}.$$

Now set $x := 1$. (According to Beckmann, Brouncker merely announced his result—the above derivation is essentially due to Euler.)

8. Consider the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots. \tag{11.1.23}$$

Observe that $\sin x = 0$ exactly when $x = \pm k\pi$. Now observe that on setting $y := x^2$,

$$1 - \frac{y}{3!} + \frac{y^2}{5!} - \frac{y^3}{7!} + \cdots = 0 \tag{11.1.24}$$

exactly when $y = (k\pi)^2$, $k = 1, 2, 3, \ldots$. If (11.1.24) were a polynomial, we would know that the sum of the reciprocals of the roots of (11.1.23) equals the negative of the coefficient of y and in general the sum of the reciprocals of the powers would be expressible in terms of the coefficients and Bernoulli numbers. Thus we would deduce that

$$\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} = \frac{1}{6} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{(k\pi)^4} = \frac{1}{90}.$$

Use the product expansion for sin [Section 2.2, Exercise 1d)] to make the above argument of Euler's rigorous. (See also Exercise 14 of Section 11.3.)

9. In computing π from (11.1.19) one must evaluate $\arctan(\frac{1}{18})$ and $\arctan(\frac{1}{57})$. Use (11.1.15) to observe that

$$\arctan\left(\frac{1}{18}\right) = 18\left(\frac{1}{325} + \frac{2}{3 \cdot 325^2} + \frac{2 \cdot 4}{3 \cdot 5(325)^3} + \cdots\right)$$

and

$$\arctan\left(\frac{1}{57}\right) = 57\left(\frac{1}{3250} + \frac{2}{3(3250)^2} + \frac{2 \cdot 4}{3 \cdot 5(3250)^3} + \cdots\right).$$

Thus terms of the second series are just decimal shifts of terms of the first series. (See Ballantine [39].) How does this affect the complexity of calculating the two arctans?

10. Prove that the number

$$0.12345678910111213 \cdots n(n+1) \cdots$$

is normal. A proof may be found in Niven [56].

11.2 *ON THE TRANSCENDENCE OF* π

The problem of "squaring the circle" is the problem of constructing a square of the same area as a given circle of radius 1, or alternatively given a line segment of unit length of constructing a segment of length $\sqrt{\pi}$. The rules of construction allow for the use of an unmarked straightedge and an unmarked compass. A more precise definition of constructible is provided in Exercise 1. In fact, the constructible numbers are exactly those numbers which can be obtained from the integers by a finite sequence of rational operations and extraction of square roots. (See Exercise 1.) Thus constructible numbers are algebraic and the transcendentality of π shows the impossibility of the problem.

The Greek notion of number, based on geometric construction, made consideration of such problems more natural than they perhaps seem today. Indeed the problem had arisen by the fifth century B.C. Anaxagoras, who died in 428 B.C., had, according to Plutarch, considered it while in jail. His contemporary, Hippocrates of Chios, the author of one of the first geometry texts, also considered the question. The other classical Greek problems of "duplicating the cube" and "trisecting the angle" also arose in this period. The "Delian problem" of duplicating the cube (in volume), so named because the oracle of Apollo at Delos had prescribed duplicating the cubical altar as a means of halting the plague of 428 B.C., is equivalent to constructing $\sqrt[3]{2}$. (The impossibility of solving these problems is also discussed in Exercise 1.)

By 414 B.C. attempts at constructing π had become so numerous that Aristrophanes refers to "circle squarers" in his play "The Birds." The term came to refer to people who attempt the impossible. However, attempting the futile is not always a waste of time. As Boyer [68, p. 71] points out: