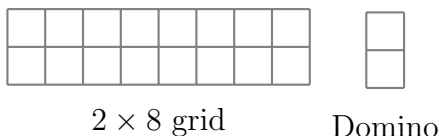# Fibonacci Numbers

The Fibonacci numbers are defined recursively by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$.

1. The Rabbit Problem: This puzzle was first proposed by Leonardo of Pisa, also known as Fibonacci, in 1202. Suppose, at the beginning of January we have a pair of baby rabbits. Rabbits cannot reproduce until they are one month old and they have a one month gestation period. Once they start reproducing, they produce a new pair of bunnies each month. Assuming that no pair ever dies, how many pairs of rabbits will there be at the end of December?

2. A drone (male) honeybee hatches from an unfertilized egg, and thus has only one parent - a female queen or worker. Female bees have two parents, a female and a drone to fertilize the egg. How many grandparents does a drone have? How many great-grandparents? How many great-great-great-great-great-grandparents?

3. How many ways are there to cover a $2 \times n$ grid with 2×1 dominoes? Prove your answer.

2×8 grid   Domino

4. Normally, people climb stairs one step at a time, but sometimes with a stretch you can skip a step. How many possible ways are there to climb a flight of n stairs, if each step can be either a single step or a skip step?

5. Prove, for any $n \in \mathbb{N}$, the Fibonacci number F_{3n} is even and F_{3n+1}, F_{3n+2} are both odd.

6. Prove, for any $n \in \mathbb{N}$, that $F_{n+5} \equiv 3F_n \pmod 5$.
   Then prove that $F_5, F_{10}, F_{15}, F_{20}, \ldots$ are all divisible by 5.

7. Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$. It starts like this:

$$a_1 = 1, \quad a_2 = \frac{1}{1+1}, \quad a_3 = \frac{1}{1 + \cfrac{1}{1+1}}, \quad a_4 = \frac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}}, \quad \ldots$$

   Prove that $a_n = \frac{F_n}{F_{n+1}}$.

8. Take an 8" by 8" sheet of gold foil, 64 inches square, and cut it up as in the picture on the left. Take the pieces and rearrange them to form the 5" by 13" rectangle on the right, giving you 65 square inches of gold foil. Now keep one square inch as profit, melt down the rest and repeat for an early retirement.