Good Problems

1. For a one form $\omega \in \mathcal{T}^1(M)$, suppose that, for all vector fields X, $\mathcal{L}_X \omega = 0$. Prove that the coefficients ω_i are constant functions in any coordinate system.

> **Solution:** Generally, $(\mathcal{L}_X\omega)(Y) = X.\omega(Y) - \omega([X,Y])$. Let $x^1, \ldots, x^n$ be local coordinates on M, and write $\omega = \sum_i \omega_i dx^i$. Using $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, we have $[X,Y] = 0$ and
>
> $$0 = (\mathcal{L}_X\omega)(Y) = X.\omega(Y) = X.\omega_j = \frac{\partial \omega_j}{\partial x^i}.$$
>
> Since ω_j has all of it's partial derivatives equal to zero, ω_j is constant.

2. Recall that M is parallelizable if the tangent bundle TM is trivial. Show that a parallelizable manifold is orientable. Give an example to show that an orientable manifold need not be parallelizable.

> **Solution:** Parallelizable means $TM = M \times \mathbb{R}^n$, or equivalently there are n non-vanishing vector fields on M which form a basis for each tangent space.
>
> The sphere S^2 is orientable but TS^2 is non-trivial - there is no non-vanishing vector field on S^2.
>
> If M is parallelizable, let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$ and let $e^1, \ldots, e^n$ be the dual basis. Then $\mu : M \to \bigwedge^n TM = M \times \bigwedge^n \mathbb{R}^n$ by $\mu(p) = (p, e^1 \wedge \cdots \wedge e^n)$ is a non-vanishing n-form on M, so M is oriented.
>
> Another approach: let $X_1, \ldots, X_n$ be non-vanishing vector fields on M which give a basis for $T_p(M)$ for all $p \in M$. Let $\omega_1, \ldots, \omega_n \in \bigwedge^1(M)$ give, at each point p, the basis of $T_p^*(M)$ dual to the X_i. Then $\mu = \omega_1 \wedge \cdots \wedge \omega_n$ is an n-form on M, and μ is nonvanishing since $\mu(X_1, \ldots, X_n) = 1$.

3. Let $\phi \in \mathcal{T}^2(M)$ be a two-tensor, and define $\phi_\Delta(X) = \phi(X,X)$. Does ϕ_Δ define a one-form?

> **Solution:** No, unless $\phi_\Delta \equiv 0$.
>
> Suppose ϕ_Δ is a one-form. For any $X \in \mathfrak{X}(M)$ and any C^∞ function f,
>
> $$f\phi_\Delta(X) = \phi_\Delta(fX) = \phi(fX, fX) = f^2\phi(X,X) = f^2\phi_\Delta(X)$$
>
> If $\phi_\Delta(X)$ is not identically zero, then there is X so that $\phi_\Delta(X) \neq 0$ at some point $p \in M$. Then $f(p) = f(p)^2$ for any smooth function f on M, including, for example $f(x) \equiv 2$. This is a contradiction.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let M be the graph of f with the metric induced from $\mathbb{R}^3$. Show that the metric volume 2-form on M is $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx \wedge dy$.

Solution: With $z = f(x,y)$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Way 1: Using the local expression for the metric volume form (Boothby pg 219/Lee Prop. 9.21)

$$dx^2+dy^2+dz^2 = \left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)dx^2+\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}dxdy+\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}dydx+\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)dy^2$$

The metric g_{ij} is given by the matrix

$$\begin{pmatrix} 1+\left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} & 1+\left(\frac{\partial f}{\partial y}\right)^2 \end{pmatrix}.$$

Then $g = \det(g_{ij}) = 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ and the volume form is $\sqrt{g}dx \wedge dy$.

Way 2: Taking the cross product of the tangent vectors $(1, 0, \frac{\partial f}{\partial x})$ and $(0, 1, \frac{\partial f}{\partial y})$ gives a normal vector field

$$N = -\frac{\partial f}{\partial x}\frac{\partial}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

and unit normal $n = \frac{N}{||N||}$. Then the volume form on M is given by

$$\begin{aligned} \iota_n(dx\wedge dy\wedge dz) &= \frac{1}{||N||}\left(-\frac{\partial f}{\partial x}dy\wedge dz + \frac{\partial f}{\partial y}dx\wedge dz + dx\wedge dy\right) \\ &= \frac{1}{||N||}\left(-\left(\frac{\partial f}{\partial x}\right)^2 dy\wedge dx + \left(\frac{\partial f}{\partial y}\right)^2 dx\wedge dy + dx\wedge dy\right) \\ &= \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\; dx\wedge dy. \end{aligned}$$

5. Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be the 'slinky curve'

$$\gamma(t) = \left(\cos(t)(\cos(10t)+2),\ \sin(t)(\cos(10t)+2),\ \frac{2t}{3}+\sin(10t)\right)$$

and let $\alpha = ydx + xdy + dz$. Compute $\int_\gamma \alpha$.

Solution: Note that $d\alpha = dy \wedge dx + dx \wedge dy = 0$, so α is closed and therefore exact. There is some $f(x, y, z)$ with $df = \alpha$. Since $\frac{\partial f}{\partial z} = 1$, $f(x, y, z) = g(x, y) + z$. Since $\frac{\partial f}{\partial x} = y$, $g(x, y) = xy + h(y)$, so $f = xy + z + h(y)$. Finally, $\frac{\partial f}{\partial y} = x$, so $h(y) = c$ for some constant c (which we set to zero) so that $f(x, y, z) = xy + z$.

Now $\gamma(0) = (3, 0, 0)$ and $\gamma(2\pi) = (3, 0, 4\pi/3)$, so that

$$\int_\gamma \alpha = \int_\gamma df = f(3, 0, 4\pi/3) - f(3, 0, 0) = 4\pi/3$$

6. Let D^n and S^{n-1} be the unit ball and unit sphere in $\mathbb{R}^n$.
   (a) With
   $$\mu = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n,$$
   show that $\mu|_{S^{n-1}}$ is the volume form on S^{n-1}.
   (b) Show that $\operatorname{vol} S^{n-1} = n \operatorname{vol} D^n$. Hint: Apply Stokes' theorem to μ integrated over S^{n-1}.

Solution:

(a) The unit normal field to S^{n-1} is given by $N = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$, and
$$\iota_N dx^1 \wedge \cdots \wedge dx^n = \mu$$

(b) Compute $d\mu = n dx^1 \wedge \cdots \wedge dx^n$. Then applying Stokes' Theorem with $S^{n-1} = \partial D^n$,
$$\operatorname{vol} S^{n-1} = \int_{S^{n-1}} \mu = \int_{D^n} d\mu = \int_{D^n} n dx^1 \wedge \cdots \wedge dx^n = n \operatorname{vol} D^n.$$

7. Let (M, g) be a Riemannian manifold with metric two-form g. A vector field $X \in \mathfrak{X}(M)$ is called a *Killing field* if $\mathcal{L}_X g = 0$. Let X be a complete Killing field with flow $\phi_t : M \to M$.

   This problem is to show that X preserves lengths of curves on M (and therefore preserves distances between points of M).

   Suppose $\gamma : [a, b] \to M$ is a smooth curve. Then $\phi_t \circ \gamma$ is also a smooth curve. Show that, for all t,
   $$\operatorname{len}(\phi_t \circ \gamma) = \operatorname{len}(\gamma)$$

   Hint: Show that the t derivative of the length vanishes.

Solution:

$$\begin{aligned}
\frac{\partial}{\partial t}\operatorname{len}(\phi_t\circ\gamma) &= \frac{\partial}{\partial t}\int_a^b \sqrt{g\big((\phi_t\circ\gamma)'(s),(\phi_t\circ\gamma)'(s)\big)}ds && (1)\\
&= \int_a^b \frac{\partial}{\partial t}\sqrt{g\big(T\phi_t\gamma'(s),T\phi_t\gamma'(s)\big)}ds && (2)\\
&= \int_a^b \frac{\partial}{\partial t}\sqrt{(\phi_t^*g)(\gamma'(s),\gamma'(s))}ds && (3)\\
&= \int_a^b \frac{1}{2\sqrt{(\phi_t^*g)(\gamma'(s),\gamma'(s))}}\frac{\partial}{\partial t}(\phi_t^*g)(\gamma'(s),\gamma'(s))ds && (4)\\
&= \int_a^b \frac{1}{2\sqrt{(\phi_t^*g)(\gamma'(s),\gamma'(s))}}(\mathcal{L}_Xg)(\gamma'(s),\gamma'(s))ds && (5)\\
&= 0 && (6)
\end{aligned}$$

So the function $\operatorname{len}(\phi_t\circ\gamma)$ has vanishing t deriviative, and is constant in t.

8. Let α,β be k-forms on a smooth n-manifold M. Let S be a k-dimensional submanifold of M (without boundary). If $[\alpha]=[\beta]\in H^k(M)$, (i.e. α and β represent the same cohomology class) then show that

$$\int_S\alpha=\int_S\beta$$

Solution: Write $\alpha=\beta+d\tau$ for some $k-1$ form τ. Then applying Stokes' Theorem,

$$\int_S\alpha=\int_S\beta+d\tau=\int_S\beta+\int_S d\tau=\int_S\beta+\int_{\partial S}\tau=\int_S\beta$$

Here are some reasonable questions from our texts, that were not already assigned.

From Lee: Ch1 Problem 19. Ch 2 Exercises 2.82, 2.90, 2.100, 2.111, 2.117. Ch2 Problems 3,4, 23, 24, 30. Ch6 Problem 1. Ch 7 Problem 6. Ch 8 Exercises 8.11, 8.18, 8.52, 8.70, 8.71, 8.72, 8.77, 8.80. Ch 8 Problems 2,4,6,9,10,11,12. Ch 9 Exercises 9.58, 9.59. Ch 9 Problem 1. Ch 10 Exercises 10.1, 10.2. Ch 10 Problems 1, 5, 7, 8, 9.

From Boothby (Chapter V): Section 1# 3,7. Section 3 # 1, 4, 7. Section 5 # 2, 6. Section 6 # 1, 2, 3, 4, 7. Section 8 # 3,5,6,7,8.