Math 642 # Good Problems

1. For a one form $\omega \in \mathcal{T}^1(M)$, suppose that, for all vector fields X, $\mathcal{L}_X\omega = 0$. Prove that the coefficients ω_i are constant functions in any coordinate system.

2. Recall that M is parallelizable if the tangent bundle TM is trivial. Show that a parallelizable manifold is orientable. Give an example to show that an orientable manifold need not be parallelizable.

3. Let $\phi \in \mathcal{T}^2(M)$ be a two-tensor, and define $\phi_\Delta(X) = \phi(X, X)$. Does ϕ_Δ define a one-form?

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let M be the graph of f with the metric induced from $\mathbb{R}^3$. Show that the metric volume 2-form on M is $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx \wedge dy$.

5. Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be the 'slinky curve'
$$\gamma(t) = \left(\cos(t)(\cos(10t) + 2),\ \sin(t)(\cos(10t) + 2),\ \frac{2t}{3} + \sin(10t)\right)$$
and let $\alpha = y dx + x dy + dz$. Compute $\int_\gamma \alpha$.

6. Let D^n and S^{n-1} be the unit ball and unit sphere in $\mathbb{R}^n$.
   (a) With
   $$\mu = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n,$$
   show that $\mu|_{S^{n-1}}$ is the volume form on S^{n-1}.
   (b) Show that $\operatorname{vol} S^{n-1} = n \operatorname{vol} D^n$. Hint: Apply Stokes' theorem to μ integrated over S^{n-1}.

7. Let (M, g) be a Riemannian manifold with metric two-form g. A vector field $X \in \mathfrak{X}(M)$ is called a *Killing field* if $\mathcal{L}_X g = 0$. Let X be a complete Killing field with flow $\phi_t : M \to M$.

   This problem is to show that X preserves lengths of curves on M (and therefore preserves distances between points of M).

   Suppose $\gamma : [a, b] \to M$ is a smooth curve. Then $\phi_t \circ \gamma$ is also a smooth curve. Show that, for all t,
   $$\operatorname{len}(\phi_t \circ \gamma) = \operatorname{len}(\gamma)$$
   Hint: Show that the t derivative of the length vanishes.

8. Let α, β be k-forms on a smooth n-manifold M. Let S be a k-dimensional submanifold of M (without boundary). If $[\alpha] = [\beta] \in H^k(M)$, (i.e. α and β represent the same cohomology class) then show that
$$\int_S \alpha = \int_S \beta$$

Here are some reasonable questions from our texts, that were not already assigned.

From Lee: Ch1 Problem 19. Ch 2 Exercises 2.82, 2.90, 2.100, 2.111, 2.117. Ch2 Problems 3,4, 23, 24, 30. Ch6 Problem 1. Ch 7 Problem 6. Ch 8 Exercises 8.11, 8.18, 8.52, 8.70, 8.71, 8.72, 8.77, 8.80. Ch 8 Problems 2,4,6,9,10,11,12. Ch 9 Exercises 9.58, 9.59. Ch 9 Problem 1. Ch 10 Exercises 10.1, 10.2. Ch 10 Problems 1, 5, 7, 8, 9.

From Boothby (Chapter V): Section 1# 3,7. Section 3 # 1, 4, 7. Section 5 # 2, 6. Section 6 # 1, 2, 3, 4, 7. Section 8 # 3,5,6,7,8.