Some Midterm Exercises

1. In a smooth m-manifold M, show every point $p \in M$ has a chart (U, φ) with $\varphi(p) = \mathbf{0}$ and $\varphi(U) = B(\mathbf{0}, 1)$, the open ball of radius 1 around $\mathbf{0} \in \mathbb{R}^m$.

Solution: Let (V, ψ) be any chart containing p. Suppose $\psi(p) = \mathbf{x}$. Let $r > 0$ so that the open ball $B(\mathbf{x}, r) \subset \psi(V)$ (ψ is a homeomorphism and V is open). Put $U = \psi^{-1}(B(\mathbf{x}, r))$ and $\varphi = \frac{1}{r}\psi - \mathbf{x}$. Then (U, φ) is the desired chart.

2. Let $f : M \to N$ be a smooth map of manifolds. Define the graph of f to be $\Gamma(f) \subset M \times N$ as $\Gamma(f) = \{(x, y) \in M \times N | f(x) = y\}$. Show $\Gamma(f)$ is a manifold.

Solution: For $x_0 \in M$, let (U, φ) be a chart for M with $x_0 \in U$, and let (V, ψ) be a chart for N with $f(x_0) = y_0 \in V$. Define $W_{x_0} = \{(x, y) \in \Gamma(f) | x \in U \text{ and } y \in V\}$. For $(x, y) \in W_{x_0}$, let $\zeta_{x_0}(x, y) = (\varphi(x), \psi(y)) \in \mathbb{R}^{m+n}$. If charts for x_1, x_2 overlap, then $\zeta_{x_2}\zeta_{x_1}^{-1} = (\varphi_{x_2}\varphi_{x_1}^{-1}, \psi_{x_2}\psi_{x_1}^{-1})$ is smooth since the component maps are smooth.

Finally, the map $\Gamma(f) \to M$ given by $(x, y) \to x$ is a homeomorphism (actually, it's a diffeomorphism), so $\Gamma(f)$ is Hausdorff and paracompact since M is.

3. Define $\sigma : M \times M \to M \times M$ by $\sigma(x, y) = (y, x)$. Show that σ is a diffeomorphism.

Solution: Fix $(x, y) \in M \times M$, and let (U, φ), (V, ψ) be charts on M containing x and y respectively. Then $(U \times V, \varphi \times \psi)$ is a chart on $M \times M$ containing (x, y), and $(V \times U, \psi \times \varphi)$ is a chart on $M \times M$ containing (y, x). In these coordinates, $T\sigma$ is given by the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ which is nonsingular. This also shows σ is smooth, since in these coordinates all partials of σ are smooth (constant!) functions. The inverse mapping theorem says σ is locally a diffeomorphism, and because σ is one-to-one and onto, σ is a diffeomorphism.

Another approach, suggested by Rob: Show σ is smooth, either by a coordinate argument as above, or by realizing σ as a compositon of other smooth maps. Then, $\sigma^{-1} = \sigma$ is also smooth, and σ is a diffeomorphism directly by the definition.

4. Given points $x_1, \ldots, x_k \in M$, and values $v_1, \ldots, v_k \in \mathbb{R}$, show there is a smooth function $f : M \to \mathbb{R}$ with $f(x_i) = v_i$ for all i.

Solution: Let U_i be an open neighborhood containing x_i so that $x_j \notin \bar{U}_i$ for $j \neq i$. Let $V_i = U_i - \bigcup_{j \neq i} \bar{U}_j$, so the V_i are disjoint open neighborhoods of the x_i. Finally, let W_i be an open neighborhood of x_i with $\bar{W}_i \subset V_i$. For each i, take a smooth cutoff function φ_i which vanishes on $V_i - W_i$ and with $\varphi_i(x_i) = 1$. Extend each φ_i (by zero) to all of M, so that in particular $\varphi_i(x_j) = 0$ for $j \neq i$. Let $f = \sum_i v_i \varphi_i$. Then f is smooth and $f(x_i) = v_i$.

5. In homogeneous coordinates on $\mathbb{R}P^1$, every point but $[1:0]$ can be written as $[x:1]$, and every point but $[0:1]$ can be written as $[1:y]$. Away from those two points, write $\frac{\partial}{\partial x}$ in terms of $\frac{\partial}{\partial y}$.

Solution: The change of coordinates is given by $x \to 1/x = y$. This has derivative $-1/x^2$. Then $\frac{\partial}{\partial x} \to \frac{-1}{x^2}\frac{\partial}{\partial y}$, or $\frac{\partial}{\partial x} = -y^2\frac{\partial}{\partial y}$.

6. On the torus $\mathbb{T}^2 = S^1 \times S^1 = \{(e^{i\theta}, e^{i\phi})\}$, define a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f(e^{i\theta}, e^{i\phi}) = (e^{i(a\theta+b\phi)}, e^{i(c\theta+d\phi)})$$

where a, b, c, d are integers. Show that f is well defined, and is a diffeomorphism if $ad - bc = \pm 1$.

Solution: To see f is well defined, let $\theta' = \theta + 2k\pi$ and $\phi' = \phi + 2\ell\pi$ Then

$$f(e^{i\theta'}, e^{i\phi'}) = (e^{i(a\theta+b\phi)+2\pi i(ak+b\ell)}, e^{i(c\theta+d\phi)+2\pi i(ck+d\ell)}) = f(e^{i\theta}, e^{i\phi})$$

since a, b, c, d are integers. Using θ, ϕ as local coordinates on $\mathbb{T}^2$, $Tf = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the specific charts used will require θ, ϕ in some range of angles, $(0, 2\pi)$ or $(-\pi, \pi)$ for example, but that the matrix form of Tf is the same in all choices. Tf is nonsingular if $ad - bc \neq 0$, and so f is a local diffeomorphism by the inverse function theorem. When $ad - bc = \pm 1$, f has an inverse given by the inverse matrix $\pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. When $ad - bc = 1$, this means $f^{-1}(e^{i\theta}, e^{i\phi}) = (e^{i(d\theta - b\phi)}, e^{i(-c\theta + a\phi)})$ and the -1 case is similar. Then f is a diffeomorphism.