Math 641 Good Problems

Questions get two ratings: A number which is relevance to the course material, a measure of how much I expect you to be prepared to do such a problem on the exam. 3 means 'of course you know this information', 1 means 'you probably need to check something in the book for this one'. Given that you know the material, the starred problems are harder.

Reasonable questions from Lee: Exercises 2.66, 2.77. Ch 2 Problems 11,13,16,23. Exercises 3.6-3.9. Ch 3 Problems 1,2,5,7,11. Exercises 4.6,4.16. Equation (4.8). Problem 4.12. Exercise 6.48.

(3) 1. Show that a connected manifold is path connected.

Solution: Pick $p \in M$. Let $C = \{x \in M | \text{There is a path between } x \text{ and } p\}$.

If $x \in C$, choose a chart (U, φ) containing x and a path c from p to x. Since $\varphi(U)$ is open in $\mathbb{R}^m$, there is $r > 0$ with the open ball $B = B(\varphi(x), r) \subset \varphi(U)$. For any $y \in \varphi^{-1}(B)$, there is a path s from $\varphi(x)$ to $\varphi(y)$ in B, so that $\varphi^{-1} \circ s$ is a path in M from x to y. Then c followed by $\varphi^{-1} \circ s$ is a path in M from p to y, so that $y \in C$. That is, C contains the open neighborhood $\varphi^{-1}(B)$ of x, and so C is open.

(2) 2. Let D be a derivation on $C^\infty(M)$. Suppose $f, g \in C^\infty(M)$, and that g is never 0. Prove the quotient rule:

$$D\left(\frac{f}{g}\right) = \frac{gDf - fDg}{g^2}$$

Solution: Since $g \neq 0$, $f/g \in C^\infty(M)$. Then by Leibniz' rule:

$$Df = D\left(g \cdot \frac{f}{g}\right) = Dg \cdot \frac{f}{g} + gD\left(\frac{f}{g}\right)$$

Solving for $D(f/g)$ gives the result.

(3) 3. Given a sequence of open sets $\{U_i\}_{n=1}^\infty$ with $\bar{U_n} \subset U_{n+1}$ for all n, and with $\cup_{i=1}^\infty U_n = M$. Say that a sequence $x_1, x_2, \ldots$ *leaves all* U if for any n there is N so that $x_i \notin U_n$ for $i > N$.

Show that there is a smooth function $f : M \to R$ so that $\lim_{i\to\infty} f(x_i) = +\infty$ for any sequence $\{x_i\}_{i=1}^\infty$ which leaves all U.

Solution: Let b_n be a cutoff function which is 1 on U_{n-1} and 0 on the complement of U_n (for $n = 1$, set $b_1 = 0$). Let $\phi_n = 1 - b_n$, so ϕ_n is 0 on U_{n-1} and 1 outside of U_n. For $x \in U_n$, there is a neighborhood $V \subset U_n$ of x, and for any $i > n$, $\phi_i \equiv 0$ on V. Define $f = \sum_{i=1}^\infty \phi_i$, which is a finite sum in a neighborhood of any x, so f is smooth.

Suppose a sequence $\{x_i\}$ leaves all U. Given $n > 0$, there is N so that $x_i \notin U_n$ for $i > N$. Then for $i > N$, $x_i \notin U_n$ so

$$f(x_i) \geq \sum_{k=1}^n \phi_k(x) = \sum_{k=1}^n 1 = n$$

which shows $f(x_i) \to \infty$.

(3) 4. Which of these homeomorhpisms are diffeomorphisms from $\mathbb{R}^2 \to \mathbb{R}^2$?

(a) $(x, y) \to (x^3, y^3)$

(b) $(x, y) \to (x^3 + x, y^3 + y)$

(c) $(x, y) \to (x\cos(x^2 + y^2) - y\sin(x^2 + y^2), x\sin(x^2 + y^2) + y\cos(x^2 + y^2))$

Solution: Parts b,c are diffeos but a is not. Part c rotates (x, y) by the angle $r^2 = x^2 + y^2$.

(**2) 5. Let $M(2)$ denote the space of 2×2 matrices with real entries. Let $N = \{A \in M(2) | A \neq 0, \det(A) = 0\}$. Show that N is a manifold.

Solution:

Way 1: Let U_ℓ be the set of matrices in N with nonzero left column, and U_r be the set of matrices in N with nonzero right column. Note that $N = U_\ell \cup U_r$. For $A \in U_\ell$, write $A = \begin{pmatrix} x & \lambda x \\ y & \lambda y \end{pmatrix}$ (which we can do because the columns of A are linearly dependent). Put $\phi_\ell(A) = (x, y, \lambda)$. Similarly, for $A \in U_r$, write $A = \begin{pmatrix} \lambda x & x \\ \lambda y & y \end{pmatrix}$ and put $\phi_r(A) = (x, y, \lambda)$. On $U_\ell \cap U_r$, the change of coordinates map is given by $(\phi_r^{-1} \circ \phi_\ell)(x, y, \lambda) = (\lambda x, \lambda y, \lambda^{-1})$, which is smooth. The inverse $\phi_\ell^{-1} \circ \phi_r$ has the same formula and is also smooth. Then (U_ℓ, φ_ℓ) and (U_r, φ_r) define an atlas on N.

Way 2: For $A \in N$, the kernel of A is a line through the origin. Let U_h be the set of $A \in N$ whose kernel is not horizontal, and U_v be the A with kernel which is not vertical. For $A \in U_h$, let $\theta \in (0, \pi)$ be the angle that $\ker A$ makes with the positive x-axis (well defined on U_h). Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation by θ. Then $AR_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$, so $AR_\theta = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ and define $\varphi_h(A) = (x, y, \theta)$. Note $\begin{pmatrix} x \\ y \end{pmatrix} = AR_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Similarly define $\varphi_v(A)$ on U_v, except $\theta \in (-\pi/2, \pi/2)$. When $\ker A$ has positive slope, $\varphi_h(A) = \varphi_v(A)$ so the coordinate change is just the identity. When $\ker A$ has negative slope, if $\varphi_h(A) = (x, y, \theta)$ then $\varphi_v(A) = (-x, -y, \theta - \pi)$ since $R_{\theta-\pi} = -R_\theta$. Then (U_h, φ_h) and (U_v, φ_v) define an atlas on N.

Note: Way 1 and way 2 are reminescent of putting stereographic and angular coordinates on a circle, respectively. In both cases, it's easy to see that the set of matrices in N with a fixed λ or θ form a two dimensional vector space, so that N is a vector bundle over the circle. N is a trivial bundle over S^1 (show it!) so that N is diffeomorphic to $\mathbb{R}^2 \times S^1$.

Bonus: Generalize these results to $N \subset M(n)$, the set of $n \times n$ matrices with one dimensional kernel. What dimension is N? Generally, N is a bundle over $\mathbb{R}P^{n-1}$ with projection $\pi : N \to \mathbb{R}P^{n-1}$ given by $\pi(A) = \ker A$. Is this a trivial bundle?

(3) 6. For a smooth map of manifolds $f : M \to N$, say that f is *self-transverse* if for all $x, y \in M$ there are neighborhoods $x \in U$, $y \in V$ so that $f|_U \pitchfork f|_V$.

(a) Give an example of M, N and $f : M \to N$ which is not self-transverse.

(b) Give an example of M, N and $f : M \to N$ which is self-transverse and not injective.

(c) Suppose $f : M \to N$ is a self-transverse immersion. Show $K = \{x \in M | \exists x' \in M \text{ with } f(x) = f(x')\}$ is a regular submanifold of M.

Except that part (c) is false! (*) Give an example to show part (c) is false.

Solution:

(a) Here are some: $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (\cos t, \sin t)$ is not self-transverse, for example between $t = 0$ and $t = 2\pi$. Any curve in $\mathbb{R}^3$ which intersects itself is not self-transverse. If M is the dijoint union of two lines, $f : M \to \mathbb{R}^2$ by $f(s) = (s, 0)$ and $f(t) = (t, t^2)$ is not self-transverse.

(b) f could map a disjoint union of two lines onto the two axes in $\mathbb{R}^2$. Or, let $f(t) = (t\cos(t), t\sin(t))$, a spiral whose $t > 0$ branch has transverse intersection with its $t < 0$ branch.

(c) Let M be the disjoint union of three copies of $\mathbb{R}^2$ and map M to the three coordinate planes in $\mathbb{R}^3$. Then M is self-transverse, but in each copy of $\mathbb{R}^2$, K is the union of the coordinate axes, which is not a manifold.

(*2) 7. Let M be a regular submanifold of N, and let X be a vector field on M. Show there is a vector field $\tilde{X}$ on N with $\tilde{X}|_M = X$.

Solution: For $p \in M$, let $(x_1, \ldots, x_n)$ be single slice coordinates on an open set $U \subset N$ with $p \in U$. So $M \cap U = \{(x_1, \ldots, x_m, 0, \ldots, 0)\} \cap U$. On $M \cap U$, write $X = \sum_{i=1}^m X_i(x_1, \ldots, x_m)\frac{\partial}{\partial x_i}$. Define a vector field on U by

$$\tilde{X}_U(x_1, \ldots, x_n) = \sum_{i=1}^m X_i(x_1, \ldots, x_m)\frac{\partial}{\partial x_i}$$

so that $\tilde{X}_U|M = X$.

Let $V = N - M$, and define $\tilde{X}_V = 0$. Now V and the collection of U as above are an open cover for M. Take a locally finite refinement of this cover, say $\{W_\alpha\}$. Each W_α is a subset of some U (or V), so each has a vector field $\tilde{X}_\alpha = \tilde{X}_U|_{W_\alpha}$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{W_\alpha\}$. Define $\tilde{X} = \sum_\alpha \varphi_\alpha \tilde{X}_\alpha$. Fix $p \in M$, if $p \in W_\alpha$ for some α, then $X_\alpha(p) = X(p)$. Therefore

$$\tilde{X}(p) = \sum_{\alpha, p \in W_\alpha} \varphi_\alpha(p)\tilde{X}_\alpha(p) = \left(\sum_{\alpha, p \in W_\alpha} \varphi_\alpha(p)\right) X(p) = X(p).$$

(2) 8. Show that the set of closed disks in $\mathbb{R}^2$ which don't contain the origin is a manifold, and show it is diffeomorphic to $S^1 \times \mathbb{R}^2$.

Solution: We can parameterize the set of closed disks by one chart with domain $H = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$, by sending $(x, y, z) \in H$ to a disk with center (x, y) and radius z. Those which don't contain the origin form a manifold because they correspond to the open set $V = \{(x, y, z)|x^2 + y^2 > z^2\} \subset H$.

Given $(e^{i\theta}, a, b) \in S^1 \times R^2$, define

$$f(e^{i\theta}, a, b) = ((e^a + e^b)\cos(\theta), (e^a + e^b)\sin(\theta), e^b) = (x, y, z) \in V$$

This map is well defined since adding 2π to θ has no effect on (x, y, z). f is smooth, one-to-one onto V, and $f^{-1}(x, y, z) = (\frac{x+iy}{\sqrt{x^2+y^2}}, \log(\sqrt{x^2 + y^2} - z), \log z)$ is also smooth.

(1) 9. Let σ be a curve (embedded 1-manifold) in $\mathbb{R}^3$, and let σ_a be the rescaled image of σ under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. For $p \in \sigma$, compute the curvature of σ_a at ap in terms of a and the curvature of σ at p.

Solution: Let $\sigma(t)$ be a unit speed parameterization with $\sigma(0) = p$. Then $\sigma_a(t) = a\sigma(t/a)$ is a unit speed parameterization of σ_a with $\sigma_a(0) = ap$. Compute the unit tangent vector and it's derivative as:

$$\sigma_a'(t) = \sigma'(t/a) \tag{1}$$

$$T_a(t) = T(t/a) \tag{2}$$

$$T_a'(t) = \frac{1}{a}T'(t/a) \tag{3}$$

Since both curves are unit speed, the curvature satisfies $\kappa_a(ap) = \frac{1}{a}\kappa(p)$.

(2) 10. Suppose M is an embedded surface in $\mathbb{R}^3$, and let N be the rescaled image of M under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. Compute the Gauss curvature $K_N(ap)$ of N at ap in terms of a and the Gauss curvature $K_M(p)$ of M at p.

Solution: Let $\sigma(t)$ be a unit speed curve in N with $\sigma(0) = ap$. Put $\tau(t) = \frac{1}{a}\sigma(at)$, a curve in M. Notice $\tau'(t) = \frac{1}{a}\sigma'(at) \cdot a = \sigma'(at)$, so τ also has unit speed, and $\tau'(0) = \sigma'(0)$. This shows the tangent planes T_pM and $T_{ap}N$ are parallel, so a unit normal vector for M at p is also a unit normal vector for N at ap. Let $\mathbf{n}$ be a unit normal field on M and also for N, which means $\mathbf{n}(ap) = \mathbf{n}(p)$.

Now compute the shape operator S_N on N in terms of S_M on M:

$$S_N(\sigma'(0)) = (\mathbf{n}\circ\sigma)'(0) = \frac{d}{dt}\mathbf{n}(a\tau(t/a))\Big|_{t=0} \tag{4}$$
$$= \mathbf{n}(\tau(t/a))\Big|_{t=0} = (\mathbf{n}\circ\tau)'(0)\cdot\frac{1}{a} \tag{5}$$
$$= \frac{1}{a}S_M(\tau'(0)) = \frac{1}{a}S_M(\sigma'(0)). \tag{6}$$

So $S_N = \frac{1}{a}S_M$ and, taking determinants, $K_N(ap) = \frac{1}{a^2}K_M(p)$. Note that this checks with the situation where M is a sphere of radius 1, where $K_M \equiv 1$, and N is a sphere of radius a with $K_N \equiv \frac{1}{a^2}$.

It is also possible to do this by showing that curvature scales by $\frac{1}{a}$ for curves, and since Gauss curvature is the product of the two principal curvatures it must scale by $\frac{1}{a^2}$.

(1*) 11. Let $c = c(s)$ be a unit speed curve in $\mathbb{R}^3$, and suppose the Frenet frame T, N, B is defined for all s. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(s,t) = c(s) + tN(s)$. Notice that for fixed s, $f(s,t)$ is the normal line to the curve at $c(s)$, and for fixed t, $f(s,t)$ is a curve 'parallel' to c at distance t.

Find all points where f fails to be an immersion.

In the case where c is a planar curve, $f : \mathbb{R}^2 \to \mathbb{R}^2$ and these points are the critical values of f.

Solution: Let κ and τ be the curvature and torsion of c, and recall $N' = -\kappa T + \tau B$.

$$\frac{\partial f}{\partial s} = c' + tN' = T - t\kappa T + t\tau B = (1 - t\kappa)T + t\tau B. \tag{7}$$
$$\frac{\partial f}{\partial t} = N. \tag{8}$$

f is an immersion except when these two vectors are dependent, which we can check with the cross product:

$$\frac{\partial f}{\partial s} \times \frac{\partial f}{\partial t} = (1 - t\kappa)B - t\tau T.$$

Since B and T are independent, this vanishes when $t\tau = 0$ and $1 - t\kappa = 0$. Since $t\kappa = 1$, neither t nor κ can vanish. Therefore, f is an immersion except when both $\tau(s) = 0$ and $t = \frac{1}{\kappa(s)}$.

Additional remark: Geometrically, $\tau = 0$ means that c is planar to 3rd order at $p = c(s)$. Normally a curve is planar only to 2nd order – see Lee, Exercise 4.7 for a Taylor expansion that shows this. The critical value is then in the plane of the curve, at $\frac{1}{\kappa}$ along the normal line from p. This is the center of curvature for the curve at p, which is the center of a circle (radius $\frac{1}{\kappa}$) that is tangent to the curve at p to order 2. When c is a plane curve, the set of critical values of f is known as the evolute of c. The Wikipedia page for evolute has a pretty animation of f as s varies.

(2) 12. Let $M(2)$ denote the vector space of 2×2 matrices. Since $M(2)$ is a vector space, the tangent space to $M(2)$ at the identity is naturally identified with $M(2)$. Let $SL(2) \subset M(2)$ be the set of matrices of with determinant 1.

(a) Show that $SL(2)$ is a manifold.

(b) What is $\dim SL(2)$?

(c) * Show that the tangent space at the identity, $T_I SL(2)$, is exactly the space of traceless matrices $\{A \in M(2) | \operatorname{tr}(A) = 0\}$.

Bonus: Do this problem for $n \times n$ matrices instead of 2×2.

Solution: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det A = ad - bc$. Then $T \det = (d, -c, -b, a)$ which has rank 1 unless $A = 0$. So any value other than 0 is a regular value for det. In particular, 1 is a regular value for det, so $SL(2)$, the set of matrices with determinant 1, is a manifold. Because $\dim M(2) = 4$ and det has rank 1, $\dim SL(2) = 3$.

Let X be a tangent vector to $SL(2)$ at the identity. Represent X by a curve $C(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in SL(2)$, with $C(0) = I$ and $C'(0) = X$. We know $\det C(t) = 1$, so take the derivative of both sides to get

$$\frac{d}{dt}(a(t)d(t) - b(t)c(t))\Big|_{t=0} = 0, \text{so} \tag{9}$$

$$a'(0)d(0) + a(0)d'(0) - b'(0)c(0) - b(0)c'(0) = 0 \tag{10}$$

Now $C(0) = I$, so $b(0) = c(0) = 0$ and $a(0) = d(0) = 1$, so we get $a'(0) + d'(0) = 0$, or that $0 = \operatorname{tr} C'(0) = \operatorname{tr} X$.

Although one can generalize the argument above for $n > 2$ using the combinatorial definition of determinant as a sum over permutations, an easier approach is to write $\det C(t)$ as a product of the eigenvalues of $C(t)$.

(*1) 13. Suppose $M \subset \mathbb{R}^3$ is a surface, and assume that for any closed curve $C : S^1 \to M$ there is a continuous unit normal field to M defined along C. Show that M is orientable.

Solution: Assume M is connected. If not, orient each component of M separately. Let $p_0 \in M$, and let N_0 be a unit normal to M at p_0. For $p \in M$, choose any smooth curve σ joining p_0 with p, and extend N_0 along σ to a unit normal vector N_p. We need to show N_p is well defined. Suppose τ is any other curve joining p_0 with p. Together, σ and τ form a closed curve C, so there is a continuous unit normal field V along C, and by replacing with $-V$ if necessary, we may assume $V = N_0$ at p_0. Then the extension of N_0 along σ agrees with V at p, and so does the extension of N_0 along τ, so N_p is well defined. Then N is a unit normal field on M and M is orientable. (This solution lacks detail, like how to extend along a curve, what if joining τ and σ isn't smooth, and explicitly showing N is continuous.)

(*2) 14. Let (X_N, Y_N) be sterographic coordinates on $S^2 - (0,0,1)$ using the north polar projection. Let (X_S, Y_S) be stereographic coordinates on $S^2 - (0,0,-1)$ using the south polar projection. Compute $[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial X_S}]$ and $[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial Y_S}]$.

Solution: The coordinate change F from north to south is given by $(X_N, Y_N) \to (X_S, Y_S) = \frac{1}{X_N^2+Y_N^2}(X_N, Y_N)$. Compute

$$TF = \frac{1}{(X_N^2+Y_N^2)^2}\begin{pmatrix} Y_N^2 - X_N^2 & -2X_NY_N \\ -2X_NY_N & X_N^2 - Y_N^2 \end{pmatrix} = \begin{pmatrix} Y_S^2 - X_S^2 & -2X_SY_S \\ -2X_SY_S & X_S^2 - Y_S^2 \end{pmatrix}.$$

Then

$$\frac{\partial}{\partial X_N} = (Y_S^2 - X_S^2)\frac{\partial}{\partial X_S} - 2X_SY_S\frac{\partial}{\partial Y_S} \tag{11}$$

$$\frac{\partial}{\partial Y_N} = -2X_SY_S\frac{\partial}{\partial X_S} + (X_S^2 - Y_S^2)\frac{\partial}{\partial Y_S} \tag{12}$$

and so

$$\left[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial X_S}\right] = 2X_S\frac{\partial}{\partial X_S} + 2Y_S\frac{\partial}{\partial Y_S} \tag{13}$$

$$\left[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial Y_S}\right] = -2Y_S\frac{\partial}{\partial X_S} + 2X_S\frac{\partial}{\partial Y_S} \tag{14}$$

The nicest expression for this result is in spherical coordinates, where

$$(\theta, \phi) \to (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi) = (x, y, z) \to \frac{\cos\phi}{1+\sin\phi}(\cos\theta, \sin\theta) = (X_S, Y_S).$$

From this, we find

$$\frac{\partial}{\partial\theta} = -Y_S\frac{\partial}{\partial X_S} + X_S\frac{\partial}{\partial Y_S} \tag{15}$$

$$\cos\phi\frac{\partial}{\partial\phi} = -X_S\frac{\partial}{\partial X_S} - Y_S\frac{\partial}{\partial Y_S} \tag{16}$$

so that

$$\left[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial X_S}\right] = -2\cos\phi\frac{\partial}{\partial\phi} \tag{17}$$

$$\left[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial Y_S}\right] = 2\frac{\partial}{\partial\theta} \tag{18}$$

(3) 15. The Whitney Embedding Theorem says that any m-manifold embeds into $\mathbb{R}^{2m}$. Give one example of an m manifold that does not embed into $\mathbb{R}^{2m-1}$.

Solution: When $m = 1$, the circle S^1 does not embed into $\mathbb{R}$. Suppose $f : S^1 \to \mathbb{R}$ is an embedding. Since S^1 is compact, there is $\theta \in S^1$ such that $f(\theta)$ is the maximum value of f. Since f is an embedding, f is a local diffeomorphism, and therefore takes a neighborhood of θ to a neighborhood of $f(\theta)$, contradicting the maximality of $f(\theta)$. So no such embedding can exists. In fact, there is not even a continuous injective map $S^1 \to \mathbb{R}$.