Math 641

Good Problems

Questions get two ratings: A number which is relevance to the course material, a measure of how much I expect you to be prepared to do such a problem on the exam. 3 means 'of course you know this information', 1 means 'you probably need to check something in the book for this one'. Given that you know the material, the starred problems are harder.

Reasonable questions from Lee: Exercises 2.66, 2.77. Ch 2 Problems 11,13,16,23. Exercises 3.6-3.9. Ch 3 Problems 1,2,5,7,11. Exercises 4.6,4.16. Equation (4.8). Problem 4.12. Exercise 6.48.

(3) 1. Show that a connected manifold is path connected.

(2) 2. Let D be a derivation on $C^\infty(M)$. Suppose $f, g \in C^\infty(M)$, and that g is never 0. Prove the quotient rule:

$$D\left(\frac{f}{g}\right) = \frac{gDf - fDg}{g^2}$$

(3) 3. Given a sequence of open sets $\{U_i\}_{n=1}^{\infty}$ with $\bar{U_n} \subset U_{n+1}$ for all n, and with $\cup_{i=1}^{\infty} U_n = M$. Say that a sequence $x_1, x_2, \ldots$ *leaves all* U if for any n there is N so that $x_i \notin U_n$ for $i > N$.

Show that there is a smooth function $f : M \to R$ so that $\lim_{i\to\infty} f(x_i) = +\infty$ for any sequence $\{x_i\}_{i=1}^{\infty}$ which leaves all U.

(3) 4. Which of these homeomorphisms are diffeomorphisms from $\mathbb{R}^2 \to \mathbb{R}^2$?

(a) $(x, y) \to (x^3, y^3)$

(b) $(x, y) \to (x^3 + x, y^3 + y)$

(c) $(x, y) \to (x\cos(x^2 + y^2) - y\sin(x^2 + y^2), x\sin(x^2 + y^2) + y\cos(x^2 + y^2))$

(**2) 5. Let $M(2)$ denote the space of 2×2 matrices with real entries. Let $N = \{A \in M(2) | A \neq 0, \det(A) = 0\}$. Show that N is a manifold.

(3) 6. For a smooth map of manifolds $f : M \to N$, say that f is *self-transverse* if for all $x, y \in M$ there are neighborhoods $x \in U$, $y \in V$ so that $f|_U \pitchfork f|_V$.

(a) Give an example of M, N and $f : M \to N$ which is not self-transverse.

(b) Give an example of M, N and $f : M \to N$ which is self-transverse and not injective.

(c) Suppose $f : M \to N$ is a self-transverse immersion. Show $K = \{x \in M | \exists x' \in M \text{ with } f(x) = f(x')\}$ is a regular submanifold of M.

Except that part (c) is false! (*) Give an example to show part (c) is false.

(*2) 7. Let M be a regular submanifold of N, and let X be a vector field on M. Show there is a vector field $\tilde{X}$ on N with $\tilde{X}|_M = X$.

(2) 8. Show that the set of closed disks in $\mathbb{R}^2$ which don't contain the origin is a manifold, and show it is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(1) 9. Let σ be a curve (embedded 1-manifold) in $\mathbb{R}^3$, and let σ_a be the rescaled image of σ under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. For $p \in \sigma$, compute the curvature of σ_a at ap in terms of a and the curvature of σ at p.

(2) 10. Suppose M is an embedded surface in $\mathbb{R}^3$, and let N be the rescaled image of M under the map $(x, y, z) \to (ax, ay, az)$, for some $a > 0$. Compute the Gauss curvature $K_N(ap)$ of N at ap in terms of a and the Gauss curvature $K_M(p)$ of M at p.

(*1) 11. Let $c = c(s)$ be a unit speed curve in $\mathbb{R}^3$, and suppose the Frenet frame T, N, B is defined for all s. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(s,t) = c(s) + tN(s)$. Notice that for fixed s, $f(s,t)$ is the normal line to the curve at $c(s)$, and for fixed t, $f(s,t)$ is a curve 'parallel' to c at distance t.

Find all points where f fails to be an immersion.

In the case where c is a planar curve, $f : \mathbb{R}^2 \to \mathbb{R}^2$ and these points are the critical values of f.

(2) 12. Let $M(2)$ denote the vector space of 2×2 matrices. Since $M(2)$ is a vector space, the tangent space to $M(2)$ at the identity is naturally identified with $M(2)$. Let $SL(2) \subset M(2)$ be the set of matrices of with determinant 1.

(a) Show that $SL(2)$ is a manifold.

(b) What is $\dim SL(2)$?

(c) * Show that the tangent space at the identity, $T_I SL(2)$, is exactly the space of traceless matrices $\{A \in M(2) | \operatorname{tr}(A) = 0\}$.

Bonus: Do this problem for $n \times n$ matrices instead of 2×2.

(*1) 13. Suppose $M \subset \mathbb{R}^3$ is a surface, and assume that for any closed curve $C : S^1 \to M$ there is a continuous unit normal field to M defined along C. Show that M is orientable.

(*2) 14. Let (X_N, Y_N) be sterographic coordinates on $S^2 - (0,0,1)$ using the north polar projection. Let (X_S, Y_S) be stereographic coordinates on $S^2 - (0,0,-1)$ using the south polar projection. Compute $[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial X_S}]$ and $[\frac{\partial}{\partial X_N}, \frac{\partial}{\partial Y_S}]$.

(3) 15. The Whitney Embedding Theorem says that any m-manifold embeds into $\mathbb{R}^{2m}$. Give one example of an m manifold that does not embed into $\mathbb{R}^{2m-1}$.