Guidelines for Written Preliminary Examinations in Mathematics
Department of Mathematics and Computer Science
Saint Louis University

The purpose of the written PhD preliminary examination is to ensure that candidates for the doctoral degree have acquired fundamental knowledge in major areas of mathematics.

Candidates must pass written exams in three of the four areas represented by the four required sequences:

1. Math 511 & 512 (Algebra)
2. Math 521 & 522, 523 or 524 (Analysis).
3. Math 531 & 532 (Topology)
4. Math 641 & 642 (Differential Geometry)

The choice of which three a candidate takes should reflect their anticipated area of dissertation research.

A student is required to take each exam within ten months of completing the course sequence to which it corresponds. However, if the sequence was completed before entering our PhD program the exam must be taken within ten months of entering the PhD program. If a student fails the exam, the student must retake the exam within a year.

The preliminary exam in each area is a 3-hour exam. It will be offered three times a year: at the beginning and end of each summer and near the beginning of the spring semester. A student who wishes to take one or more preliminary exams should make a formal request to the Director of Graduate Studies. The deadlines for these requests are April 15, July 15 and November 15 respectively. Exams will not be offered at any other times during the year.

The Director of Graduate Studies will appoint one member of the Graduate Faculty to prepare and grade each student's exam and one other member of the Graduate Faculty to assist in this task. In most cases the faculty member who taught the student part or all the graduate level sequence will be the faculty member in charge of the exam. Sample preliminary exams from previous years are available in the Math Department office.

If a student fails any area's exam twice, the student may not continue in the PhD program.

January 2009

Differential Geometry Qualifying Exam
June, 2010

INSTRUCTIONS: Do any seven of the ten problems.

1. In $\mathbb{R}^3$, set

$$X = x^2y\,\frac{\partial}{\partial y} - x\,\frac{\partial}{\partial z}, \quad Y = 2x\,\frac{\partial}{\partial x}, \quad \omega = z\,dx \wedge dy + y^2\,dx \wedge dz$$

Compute:

   (a) $[X, Y]$
   (b) $\omega(X, Y)$
   (c) $\omega \wedge (y\,dy)$
   (d) $d\omega$

2. Let M be a smooth 2–dimensional manifold, and let (U, φ) and (U, ψ) be two smooth charts on M with the same domain. Assume that the change of coordinates $\psi \circ \varphi^{-1}$ is given by the formula $\psi \circ \varphi^{-1}(x, y) = (x^2 - y^2, 2xy)$ and that it carries the first quadrant $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ onto the upper half–plane $\{(u, v) \in \mathbb{R}^2 : v > 0\}$.

   (a) If a one–form has the local expression vdu in the chart (U, ψ), find its local expression in the chart (U, φ).
   (b) If a vector field has the local expression $\frac{\partial}{\partial x}$ in the chart (U, φ), find its local expression in the chart (U, ψ).

3. Prove that the composition of two smooth embeddings is a smooth embedding.

4. Let ∇ be a linear connection on a manifold M with non–vanishing torsion T. For X, Y in $\mathfrak{X}(M)$, define

$$\nabla^*_X Y = \nabla_X Y - T(X, Y).$$

Show that ∇^* is a linear connection with torsion $-T$.

5. Explain what a Riemannian metric is, and show that every smooth manifold admits a Riemannian metric.

6. Let $\alpha : S^n \to S^n$ denote the antipodal map, and let $q : S^n \to \mathbb{R}P^n$ denote the quotient map. Suppose $\eta \in \Omega^1(S^n)$. Show there exists an $\omega \in \Omega^1(\mathbb{R}P^n)$ such that $\eta = q^*\omega$ if and only if $\alpha^*\eta = \eta$.

7. Prove that the product manifold $M \times N$ is orientable if and only if both M and N are orientable.

8. Let M be a manifold with a linear connection ∇, and let X be a vector field on M. Given a 1–form $\alpha \in \Omega^1(M)$, define $\nabla_X\alpha : \mathfrak{X}(M) \to C^\infty(M)$ by

$$[\nabla_X\alpha](Z) = X(\alpha(Z)) - \alpha(\nabla_X Z) \quad \text{for} \quad Z \in \mathfrak{X}(M).$$

   (a) Show that $\nabla_X\alpha$ is a 1–form on M.

   (b) Consider $\nabla_X : \Omega^1(M) \to \Omega^1(M)$ as established in (a). Prove that

$$\nabla_X(f\alpha) = X(f)\alpha + f\nabla_X\alpha$$

   for all $f \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$.

9. Define a Lie group G by

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

   (a) Find a basis for the left invariant vector fields on G expressed in the x, y coordinate system.

   (b) Find a basis for the left–invariant 1–forms on G expressed in the x, y coordinate system.

10. Let G be the Lie group in the previous problem (9).

   (a) Identify the Lie algebra of G with a Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{gl}(3, \mathbb{R})$ of 3×3 real matrices.

   (b) Compute a simple formula for the exponential map $\exp : \mathfrak{g} \to G$.

Ph.D. Exam
Differential Geometry
August 2005

Do any six problems.

1. Define $f : \mathbf{R}^3 \to \mathbf{R}^2$ by $f(x,y,z) = (x^2 + y^2, yz)$. Let (u,v) denote standard coordinates in $\mathbf{R}^2$.
   a. Calculate $f^*(udv + vdu)$.
   b. Calculate $f_*\left(\frac{\partial}{\partial y}\Big|_{(1,3,-1)}\right)$.
   c. Find a regular value of f.

2. Suppose $f : M \to N$ is a one–to–one immersion. Prove that f is an embedding if M is compact.

3. Let X and Y be two smooth vector fields on a smooth manifold M. Suppose that $\omega(X) = \omega(Y)$ for all smooth 1–forms on M. Show that $X = Y$.

4. Let T be a tensor field of type $(2,0)$ and let X be a vector field. Define

$$S(Y,Z) = X(T(Y,Z)) - T([X,Y],Z) - T(Y,[X,Z])$$

for all vector fields Y, Z. Prove that S is a tensor field of type $(2,0)$.

5. Show that if H_1 and H_2 are Lie subgroups of G then $H_1 \cap H_2$ is a Lie subgroup of G. Hint: What should the Lie algebra of this intersection be? Be careful: $H_1 \cap H_2$ may have countably many distinct components even if H_1 and H_2 are connected. You may find it helpful to recall that the exponential map of G carries a neighborhood of 0 in $\mathcal{G}$ diffeomorphically onto a neighborhood of e in G.

6. Let M be an oriented compact n–dimensional manifold (without boundary). Let ω be a p–form and η an $(n-p-1)$–form on M. Show that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

7. Describe the smooth atlas for the tangent bundle TM of a smooth m–dimensional manifold M. Calculate the change of charts for this atlas.

DIFFERENTIAL GEOMETRY PRELIMINARY EXAMINATION

August 23, 2002

* **Direction:** Do any 5 problems.

1. Let M be a differentiable manifold, U an open subset of M, and C a closed subset of M with $C \subset U$. Show that if X is a differentiable vector field on U, then $X|_C$, the restriction of X on C, extends to a differentiable vector field $\widetilde{X}$ on M. Can X be extended to a vector field on M? Prove or disprove.

2. Let $\Phi : GL(n) \to GL(n)$ be the smooth mapping given by $\Phi(A) = A^T \cdot A$ for $A \in GL(n)$.
   (a) Prove that relative to the standard identification $T_{I_n}(GL(n)) = M(n)$, the differential $d\Phi_{I_n} : T_{I_n}(GL(n)) \to T_{I_n}(GL(n))$ has the formula $d\Phi_{I_n}(A) = A^T + A$.
   (b) Show that the map Φ has constant rank $\dfrac{n(n+1)}{2}$ on $GL(n)$.
   (c) Show that the orthogonal subgroup $O(n)$ of $GL(n)$ is a smooth, compact submanifold of the manifold $GL(n)$ of dimension $\dfrac{n(n-1)}{2}$.

3. On $\mathbf{R}^2 - \{0\}$ consider the differential 2-form
   $$\omega(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x dy \wedge dz - y dx \wedge dz + z dx \wedge dy).$$
   (a) Show that ω is a closed 2-form on $\mathbf{R}^2 - \{0\}$.
   (b) Show that $\displaystyle\int_{S^2} i^*\omega = 4\pi$. (Suggestion: Use polar coordinates on S^2)
   (c) Show that $[\omega] \neq 0$ in $H^2(\mathbf{R}^2 - \{0\})$.

4. Let D be the Euclidean connection on $\mathbf{R}^2$, and let X and Y be vector fields on $\mathbf{R}^2$ given by $X = (x^2 y)\dfrac{\partial}{\partial x} + y^2 \dfrac{\partial}{\partial y}$ and $Y = x^2 \dfrac{\partial}{\partial x} - (x^3 y^3)\dfrac{\partial}{\partial y}$. Compute $D_X Y$, and $(D_X Y)(p)$ where $p = (1, -1)$.

5. Let M be a differentiable 2-manifold in $\mathbf{R}^3$ and let $\alpha : [a,b] \to M$ be a differentialble curve in M. Show that α is a geodesic in M with respect to the Levi-Civita connection on M if and only if $\alpha''(t) \perp M$, i.e. $\alpha''(t) \perp T_{\alpha(t)}(M)$ for all $t \in [a,b]$.

(Continue to the back page)

6. Consider the Poincaré upper half-plane $\mathbf{H}^2 = \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$ with the hyperbolic metric $g = \dfrac{1}{y^2}(dx \otimes dx + dy \otimes dy)$.
   (a) Calculate the matrix representation $[g_{ij}]_{2\times 2}$ of the metric g.
   (b) Compute the Christoffel symbols Γ^k_{ij} $(i,j,k = 1,2)$ for the Levi-Civita connection on $\mathbf{H}^2$.

7. Consider the semi-Riemannian manifold $(\mathbf{R}^3, g)$, where $g = dx \otimes dx + dy \otimes dy - dz \otimes dz$ is the Minkowski metric on $\mathbf{R}^3$. Let $M = \{(x,y,z) \in \mathbf{R}^3 \mid x^2 + y^2 - z^2 = -1, z > 0\}$.
   (a) Show that M is a submanifold of $\mathbf{R}^3$.
   (b) Show that $(M, g|_M)$ is a Riemannian manifold, where $g|_M = i^*(g)$ and $i : M \to \mathbf{R}^3$ is the inclusion map.

PH. D. EXAM IN DIFFERENTIAL GEOMETRY
MAY, 2002

Doing 5 of the 7 problems well is sufficient.

1. Let $f : M \to N$ be a smooth map between smooth manifolds. Define $\mathrm{Gr}(f) = \{(x, y) \in M \times N \mid y = f(x)\}$. Show that $\mathrm{Gr}(f)$ is a smooth manifold.

2. Let (M, g) and (N, h) be Riemannian manifolds with differentiable atlases, respectively, of $\mathcal{A}$ and $\mathcal{B}$.

a) Show that the manifold $M \times N$ has a differentiable structure coming from $\mathcal{A}$ and $\mathcal{B}$.

b) Show that $M \times N$ has a Riemannian metric coming from g and h. (Show all that needs to be shown.)

3. Derive the Jacobi equation for a variation vector field associated to a family of geodesics $\{\gamma_u : [a, b] \to M \mid -\epsilon < u < \epsilon\}$ (M a Riemannian manifold). (Let's say that you remember it has something to do with $R(U, T)T$ and $\nabla_T^2 U$, but you can't recall exactly what it is.)

Let T be the velocity vector field, $T_{(t,u)} = \dot{\gamma}_u(t)$, and let U be the variation vector field $U_{(t,u)} = (\partial/\partial u)\gamma_u(t)$.

a) What do you know about $\nabla_T T$?

b) This being true for all u, what do you know about $\nabla_U \nabla_T T$?

c) Derive some equation involving $R(U, T)T$ and $\nabla_T^2 U$.

4. Let $M = \mathbb{C} \cup \{\infty\}$ (thought of as the complexes plus one other point). Let $U = \mathbb{C}$ and $V = (\mathbb{C} - \{0\}) \cup \{\infty\}$. Define $\phi : U \to \mathbb{R}^2$ and $\psi : V \to \mathbb{R}^2$ by

$$\begin{aligned} \phi(\zeta) &= (\mathrm{Re}(\zeta), \mathrm{Im}(\zeta)) \\ \psi(\zeta) &= (\mathrm{Re}(1/\zeta), \mathrm{Im}(1/\zeta)) \quad \text{for } \zeta \neq \infty \\ \psi(\infty) &= (0, 0). \end{aligned}$$

a) Show that $\{\phi, \psi\}$ is an atlas making M a differentiable manifold.

b) Show that complex multiplication by a constant $a + bi$ (taking ∞ to ∞) is a diffeomorphism on M.

5. Let the real plane $\mathbb{R}^2$ be parametrized by the standard rectangular coordinates (x, y). Let V and W be vector fields given by

$$\begin{aligned} V &= (x^3 y^3)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \text{and} \\ W &= \frac{\partial}{\partial x} + (x^4 y^4)\frac{\partial}{\partial y}. \end{aligned}$$

Find $\nabla_V W$ (standard covariant derivative in the plane) and $[V, W]$.

6. Let M be a surface of revolution in $\mathbb{R}^3$, parametrized by

$$F:\ (r,\theta) \mapsto (r\cos\theta,\ r\sin\theta;\ r^4).$$

Let R and Θ be the corresponding coordinate vector fields. Use the Euclidean metric and the accompanying Riemannian connection for the following:

a) Find expressions for R and Θ in terms of the standard $\mathbb{R}^3$ coordinates; find $\langle\Theta,\Theta\rangle$, $\langle\Theta,R\rangle$, and $\langle R,R\rangle$.

b) Show (such as by using part (a) and your knowledge of $[\Theta,R]$)

$$\nabla_\Theta\Theta = -\frac{r}{1+16r^6}R \quad \text{and}$$
$$\nabla_\Theta R = \frac{1}{r}\Theta.$$

c) Derive differential equations for parallel transport of a vector along an integral curve of Θ, $\sigma(\theta) = F(r_0,\theta)$ for fixed r_0. Do this by cosidering a vector field $Y_\theta = a(\theta)\Theta + b(\theta)R$ at $\sigma(\theta)$.

7. a) Let A be a (1,1) tensor field (so if X is a vector, then so is $A(X)$). For any vector field Z, define $\mathcal{L}_Z A$ as acting on vector fields by

$$(\mathcal{L}_Z A)(X) = [Z, A(X)] - A([Z,X]).$$

i. Prove that for any vector field Z, $\mathcal{L}_Z A$ is a (1,1) tensor field.
ii. Prove that the operator B, operating on pairs of vector fields by $B:(Z,X)\mapsto(\mathcal{L}_Z A)(X)$, is not tensorial.

b) Let ω be a 1-form. Define $d\omega$ operating on pairs of vector fields by

$$(d\omega)(X,Y) = X(\omega Y) - Y(\omega X) - \omega[X,Y].$$

Prove $d\omega$ is a tensor field.

Ph. D. Exam in Differential Geometry
August, 2001

Doing 5 of the 7 problems well is sufficient.

1. Let $f : M \to N$ be a smooth map between smooth manifolds. Define $\mathrm{Gr}(f) = \{(x, y) \in M \times N \mid y = f(x)\}$. Show that $\mathrm{Gr}(f)$ is a smooth manifold.

2. Let (M, g) and (N, h) be Riemannian manifolds with differentiable atlases, respectively, of $\mathcal{A}$ and $\mathcal{B}$.

a) Show that the manifold $M \times N$ has a differentiable structure coming from $\mathcal{A}$ and $\mathcal{B}$.

b) Show that $M \times N$ has a Riemannian metric coming from g and h. (Show all that needs to be shown.)

3. Derive the Jacobi equation for a variation vector field associated to a family of geodesics $\{\gamma_u : [a, b] \to M \mid -\epsilon < u < \epsilon\}$ (M a Riemannian manifold). (Let's say that you remember it has something to do with $R(U,T)T$ and $\nabla_T^2 U$, but you can't recall exactly what it is.)

Let T be the velocity vector field, $T_{(t,u)} = \dot{\gamma}_u(t)$, and let U be the variation vector field $U_{(t,u)} = (\partial/\partial u)\gamma_u(t)$.

a) What do you know about $\nabla_T T$?

b) This being true for all u, what do you know about $\nabla_U \nabla_T T$?

c) Derive some equation involving $R(U,T)T$ and $\nabla_T^2 U$.

4. Let $M = \mathbb{C} \cup \{\infty\}$ (thought of as the complexes plus one other point). Let $U = \mathbb{C}$ and $V = (\mathbb{C} - \{0\}) \cup \{\infty\}$. Define $\phi : U \to \mathbb{R}^2$ and $\psi : V \to \mathbb{R}^2$ by

$$\begin{aligned} \phi(\zeta) &= (\mathrm{Re}(\zeta), \mathrm{Im}(\zeta)) \\ \psi(\zeta) &= (\mathrm{Re}(1/\zeta), \mathrm{Im}(1/\zeta)) \quad \text{for } \zeta \neq \infty \\ \psi(\infty) &= (0,0) \end{aligned}$$

a) Show that $\{\phi, \psi\}$ is an atlas making M a differentiable manifold.

b) Show that complex multiplication by a constant $a + bi$ (taking ∞ to ∞) is a diffeomorphism on M.

5. Let the real plane $\mathbb{R}^2$ be parametrized by the standard rectangular coordinates (x, y). Let V and W be vector fields given by

$$V = (x^3y^3)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \text{and}$$
$$W = \frac{\partial}{\partial x} + (x^4y^4)\frac{\partial}{\partial y}.$$

Find $\nabla_V W$ (standard covariant derivative in the plane) and $[V, W]$.

6. Let M be a surface of revolution in $\mathbb{R}^3$, parametrized by

$$F: (r,\theta) \mapsto (r\cos\theta,\ r\sin\theta,\ r^4).$$

Let R and Θ be the corresponding coordinate vector fields. Use the Euclidean metric and the accompanying Riemannian connection for the following:

a) Find expressions for R and Θ and in terms of the standard $\mathbb{R}^3$ coordinates; find $\langle\Theta,\Theta\rangle$, $\langle\Theta,R\rangle$, and $\langle R,R\rangle$.

b) Show (such as by using part (a) and your knowledge of $[\Theta,R]$)

$$\nabla_\Theta\Theta = -\frac{r}{1+16x^6}R \quad \text{and}$$
$$\nabla_\Theta R = \frac{1}{r}\Theta.$$

c) Derive differential equations for parallel transport of a vector along an integral curve of Θ, $\sigma(\theta) = F(r_0,\theta)$ for fixed r_0. Do this by cosidering a vector field $Y_\theta = a(\theta)\Theta + b(\theta)R$ at $\sigma(\theta)$.

7. a) Let A be a (1,1) tensor field (so if X is a vector, then so is $A(X)$). For any vector field Z, define $\mathcal{L}_Z A$ as acting on vector fields by

$$(\mathcal{L}_Z A)(X) = [Z, A(X)] - A([Z,X]).$$

i. Prove that for any vector field Z, $\mathcal{L}_Z A$ is a (1,1) tensor field.
ii. Prove that the operator B, operating on pairs of vector fields by $B: (Z,X) \mapsto (\mathcal{L}_Z A)(X)$, is not tensorial.

b) Let ω be a 1-form. Define $d\omega$ operating on pairs of vector fields by

$$(d\omega)(X,Y) = X(\omega Y) - Y(\omega X) - \omega[X,Y].$$

Prove $d\omega$ is a tensor field.

Ph.D. Exam
Differential Geometry
August 2001

Do any five problems.

1. Let M be an oriented compact n–dimensional manifold (without boundary). Let ω be a p–form and η an $(n-p-1)$–form on M. Show that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

2. Let $\omega = M(x,y)dx + N(x,y)dy$ be a one form on $\mathbf{R}^2$. Show there exists a smooth function F on $\mathbf{R}^2$ such that $dF = \omega$ if and only if $d\omega = 0$.

3. What is the meaning (definition) of the Lie derivative $L_X T$ where X is a vector field and T is a (p,q)–tensor field?

4. Let $f : M \to N$ be a C^∞ map. Show that the graph of f

$$\Gamma = \{(x,y) \in M \times N : f(x) = y\}$$

is a smooth, closed embedded submanifold of $M \times N$.

5. Consider the Lie group

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & \frac{1}{x} \end{bmatrix} : x, y \in \mathbf{R}, x \neq 0 \right\}.$$

Regard (x,y) as a globally defined local coordinate system on G.
i) Exhibit the Lie Algebra $\mathcal{G}$ of G as a Lie sub–algebra of $\mathcal{GL}(2, \mathbf{R}^2)$.
ii) Find the local coordinate expressions for a basis of the left invariant vector fields on G.

6. Prove that $4x^3 + 2xy + z = 0$ is a smooth surface M in $\mathbf{R}^3$, and find a basis $\mathcal{B}$ for $T_{(0,1,0)}M$ as a vector subspace of $T_{(0,1,0)}\mathbf{R}^3 \approx \mathbf{R}^3$. Let $f : M \to \mathbf{R}^2$ be defined by $f(x,y,z) = (y,z)$. Compute the matrix of $f_* : T_{(0,1,0)}M \to T_{(0,0)}\mathbf{R}^2$ relative to the basis $\mathcal{B}$ for $T_{(0,1,0)}M$ and the basis $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ for $T_{(0,0)}\mathbf{R}^2$.

7. Describe the smooth atlas for the tangent bundle TM of a smooth m–dimensional manifold M. Calculate the change of charts for this atlas.

Ph.D. Exam
Differential Geometry
August 2000

Do any five problems.

1. Let M be an oriented compact n–dimensional manifold (without boundary). Let ω be a p–form and η an $(n-p-1)$–form on M. Show that

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

2. Let $\omega = M(x,y)dx + N(x,y)dy$ be a one form on $\mathbf{R}^2$. Show there exists a smooth function F on $\mathbf{R}^2$ such that $dF = \omega$ if and only if $d\omega = 0$.

3. What is the meaning (definition) of the Lie derivative $L_X T$ where X is a vector field and T is a (p,q)–tensor field?

4. Let $f : M \to N$ be a C^∞ map. Show that the graph of f

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7. Describe the smooth atlas for the tangent bundle TM of a smooth m–dimensional manifold M. Calculate the change of charts for this atlas.

Ph. D. Exam in Differential Geometry
1996

1. Let the real plane $\mathbb{R}^2$ be parametrized by the standard rectangular coordinates (x, y). Let V and W be vector fields given by

$$V = (x^2 y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \text{ and}$$
$$W = \frac{\partial}{\partial x} + (\cos x \sin y)\frac{\partial}{\partial y}.$$

Find $\nabla_V W$ and $[V, W]$.

2. Let M and N be manifolds with differentiable atlases, respectively, of $\mathcal{A}$ and $\mathcal{B}$, and with Riemannian metrics, respectively, of g and h.

a) Show that the manifold $M \times N$ has a differentiable structure coming from $\mathcal{A}$ and $\mathcal{B}$.

b) Show that $M \times N$ has a Riemannian metric coming from g and h. (Show all that needs to be shown.)

c) Make a reasonable guess about how the covariant derivative from the product metric works (you needn't prove it), and use that to show a curve in $M \times N$ is a geodesic if and only if each of its projections is a geodesic.

3. Let $\sigma : (a, b) \to M$ be a curve in a pseudo-Riemannian manifold.

a) If σ is reparametrized to $\bar{\sigma} : (\bar{a}, \bar{b}) \to M$, how is $\nabla_{\dot{\bar{\sigma}}}\dot{\bar{\sigma}}$ related to $\nabla_{\dot{\sigma}}\dot{\sigma}$?

b) Show that if for some scalar function $\lambda : (a, b) \to \mathbb{R}$, $\nabla_{\dot{\sigma}}\dot{\sigma} = \lambda\dot{\sigma}$, then there is a reparametrization $\bar{\sigma}$ of σ which is a geodesic.

4. For each u, let σ_u be a a unit-speed geodesic. Let $T = \dot{\sigma}_u$ and $U = \frac{\partial}{\partial u}\sigma_u$. (More precisely: For $\alpha(t, u) = \sigma_u(t)$, $T = \alpha_* \frac{\partial}{\partial t}$ and $U = \alpha_* \frac{\partial}{\partial u}$.) Derive the Jacobi equation:

$$\nabla_T \nabla_T U + R(U, T)T = 0$$

5. Consider the unit 2-sphere in $\mathbb{R}^3$, parametrized by

$$F : (\theta, \phi) \mapsto (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi).$$

Let Θ and Φ be the corresponding coordinate vector fields.

a) Find expressions for Θ and Φ in terms of the standard $\mathbb{R}^3$ coordinates; find $\langle\Theta, \Theta\rangle$, $\langle\Theta, \Phi\rangle$, and $\langle\Phi, \Phi\rangle$.

b) Use coordinate expressions to show $\nabla_\Theta \Theta = (\sin\phi\cos\phi)\Phi$. (Recall that $\nabla_X Y = \pi(D_X Y)$, where D is the covariant derivative in $\mathbb{R}^3$ and π is projection to the surface, $\pi(X) = X - \langle X, N\rangle N$, for N the unit normal vector to the surface.)

c) Use your knowledge of $\langle\Theta, \Theta\rangle$, $\langle\Theta, \Phi\rangle$, $\langle\Phi, \Phi\rangle$, and $[\Theta, \Phi]$ to show $\nabla_\Theta \Phi = (-\tan\phi)\Phi$. (Hint: Look at $\langle\nabla_\Theta\Phi, \Theta\rangle$ and $\langle\nabla_\Theta\Phi, \Phi\rangle$.)

d) Use geometric insight to show $\nabla_\Phi \Phi = 0$.

e) Use b) and c) to derive differential equations for parallel transport of a vector along an integral curve of Θ: $\sigma(t) = F(t, \phi_0)$. What is the parallel transport of Φ once around the sphere along such a curve at a given ϕ_0?

DIFFERENTIAL GEOMETRY PRELIMINARY EXAMINATION
June 17, 1996

* **Direction:** Do any 5 problems.

1. Let $M = \mathbf{C} \cup \{\infty\}$ ($\mathbf{C}$ is the set of complex numbers). Set $U = \mathbf{C}$ and $V = M - \{0\}$, and define $\varphi : U \to \mathbf{C}(=\mathbf{R}^2)$ by $\varphi(z) = z$ and $\psi : V \to \mathbf{C}(=\mathbf{R}^2)$ by $\psi(z) = 1/z$ if $z \neq 0$ and $\psi(z) = 0$ if $z = \infty$.
   (a) Show that $\{(U,\varphi), (V,\psi)\}$ defines a smooth atlas on M.
   (b) Let $p(z)$ and $q(z)$ be a pair of relatively prime polynomials with complex coefficients. Define $f : M \to M$ by $f(z) = p(z)/q(z)$ if $z \neq \infty$ and $q(z) \neq 0$, $f(z) = \infty$ if $q(z) = 0$, and $f(z) = \lim p(\omega)/q(\omega)$ if $z = \infty$. Show that f is a smooth mapping.
   (c) Show that M is diffeomorphic to the 2-sphere $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$.

2. Let U be an open subset of $\mathbf{R}^k$ and $f : U \to \mathbf{R}^{n-k}$ a smooth mapping. Show that the graph of f, $G_f = \{(x,f(x)) \in \mathbf{R}^n \mid x \in U\}$, is a smooth submanifold of $\mathbf{R}^n$ of dimension k.

3. Let $\Phi : Gl(n) \to Gl(n)$ be the smooth mapping given by $\Phi(Y) = Y^T Y$.
   (a) prove that relative to the standard identification $T_I(Gl(n)) = M(n)$, the differential $d\Phi_I : T_I(Gl(n)) \to T_I(Gl(n))$ has the formula $d\Phi_I(A) = A^T + A$.
   (b) Show that the map Φ has constant rank $n(n+1)/2$.
   (c) Show that the orthogonal group $O(n) \subset Gl(n)$ is a smooth, compact submanifold of dimension $n(n-1)/2$.
   (d) Show that the vector subspace $T_I(O(n)) \subset M(n)$ is the space of skew symmetric matrices.

4. Let $\mathbf{U}$ be the usual or canonical C^∞ structure on $\mathbf{R}$, $\Phi : \mathbf{R} \to \mathbf{R}$ the homeomorphism $\Phi(x) = x^3$, and $\mathbf{U}_\Phi = \{ (\Phi^{-1}(U_\alpha), \varphi_\alpha \circ \Phi) \mid (U_\alpha,\varphi_\alpha) \in \mathbf{U}\}$ the C^∞ structure on $\mathbf{R}$ induced by Φ. Show that the identity map $id : (\mathbf{R},\mathbf{U}_\Phi) \to (\mathbf{R},\mathbf{U})$ is not a diffeomorphism.

5. Let M be a smooth manifold, $C \subseteq U \subseteq M$ where C is clsed and U is open in M, and $f : U \to \mathbf{R}$ a smooth function.
   (a) Show that $f|_C$ can be extended to a smooth function on M.
   (b) Can f be extended to a smooth function on M? Prove or disprove.

6. Let M be a smooth manifold. Prove that a k-plane bundle over M, $\pi : E \to M$, is a trivial bundle if and only if there are (smooth) sections $s_1, \ldots, s_k \in \Gamma(E)$ such that $\{s_1(x), \ldots, s_k(x)\}$ is a basis of $E_x = \pi^{-1}(x)$ $\forall\, x \in M$.

7. Given the 1-form $\omega = x^2y\, dx + x\, dy$ on $\mathbf{R}^2$.
   (a) Evaluate the line integral $\int_s \omega$, where s is the (smooth) radial path in $\mathbf{R}^2$ from (0,0) to (1,1).
   (b) Evaluate the line integral $\int_s \omega$, where s is the piecewise smooth path in $\mathbf{R}^2$ consisting of line segments from (0,0) to (1,0) and (1,0) to (1,1).
   (c) Determine whether ω has the path independent line integrals.

8. Consider the canonical projection $\pi : \mathbf{S}^n \to \mathbf{RP}^n$ and the antipodal map $\alpha : \mathbf{S}^n \to \mathbf{S}^n$ with $\alpha(x) = -x$. Let ω be a 1-form on $\mathbf{S}^n$. Show that there exists a 1-form ω_0 on $\mathbf{RP}^n$ such that $\omega = \pi^*(\omega_0)$ if ana only if $\alpha^*(\omega) = \omega$.

Differential Geometry Comprehensive Examination

December 15, 1994

Direction: Do all seven problems. Each problem is worth 20 points.

1. Let X be a topological space and G a topological group acting continuously on X. Show that $\forall x \in X$, G_x is a closed subgroup of G.

2. Let M be a C^∞ manifold, K a compact subset of M, and U an open subset of M containing K. Prove that there is a C^∞ function $f : M \to [0,1]$ such that $f(K) \equiv 1$ and $\text{supp}(f) \subset U$.

3. (a) Let M and N be C^∞ manifolds. Show that $\forall n_0 \in N$, the map $i : M \to M \times N$ given by $i(m) = (m, n_0)$ is an imbedding.

   (b) Show that the function $F : R \to R^2$ given by $F(t) = (2\cos(t - \pi/2), \sin 2(t - \pi/2))$ is an immersion but not an imbedding.

4. (a) Show that if M is a C^∞ manifold, then $\forall X, Y \in \mathfrak{X}(M)$ and $\forall f, g \in C^\infty(M)$, $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.

   (b) Given the vector fields $X = y\,\partial/\partial x - x\,\partial/\partial y$ and $Y = z\,\partial/\partial y - y\,\partial/\partial z$ on R^3 (with coordinates x, y, z), compute the components of $[X,Y]$.

5. Let $f : R^3 \to R$ be given by $f(x,y,z) = xy + xz + yz - 1$.

   (a) Show that $f^{-1}(0)$ is a closed, regular submanifold of R^3.

   (b) Compute $f^*(dt)$.

6. Let $M = \{ (x,y,z) \in R^3 \mid x^4 + y^4 + z^4 = 1 \}$.

   (a) Give an explicit basis for $T_p(M)$ at $p = (x, y, z)$ with $z > 0$.

   (b) Show that $n = (x^3/(1 - x^4 - y^4)^{3/4}, y^3/(1 - x^4 - y^4)^{3/4}, 1)$ is a normal vector to $T_p(M)$ at p in (a).

   (c) Define $\pi : T_p(R^3) \to T_p(M)$ to be the orthogonal projection along the vector n in (b). Express $\pi(1,0,0)$ in terms of the basis for $T_p(M)$ in (a).

7. Show that if M_1 and M_2 are Riemannian manifolds with Riemannian metrics Φ_1 and Φ_2 resp., and $F : M_1 \to M_2$ is an isometry w.r.t. the Reimannian metrics Φ_1 and Φ_2, then F is an isometry of the manifolds M_1 and M_2 w.r.t. the induced metrics d_{Φ_1} and d_{Φ_2} on M_1 and M_2 respectively.

8. Let G be a Lie group and let H be an algebraic sugbroup of G. Prove that if H is normal, then $\overline{H}$ is normal. $gHg^{-1} = H$

9. (a) Let M be a smooth manifold. Let $p \in M$. Define $T_p(M)$, the tangent space of M at p.
   (b) Let M and N be smooth manifolds and let $F: M \to N$ be smooth. Define $F_{*p}: T_p(M) \to T_{F(p)}(M)$, the derivative of F at p.

10. Let $f: C \to C$ be a complex analytic (holomorphic) function. Let $z_0 \in C = R^2$. Prove that f is a submersion at z_0 if and only if $f'(z_0) \neq 0$. (Here, $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in C$.)

[<u>Hint</u>: Recall that if $f = u+iv$, then $f' = \partial u/\partial x + i\partial v/\partial x$ and $\partial u/\partial x = \partial v/\partial y, \partial u/\partial y = -\partial v/\partial x$.]

Conclude that if f is a polynomial with complex coefficients, then f is a submersion except at finitely many points.

Differential Geometry Comprehensive Examination 7/1/9

Direction: Do all seven problems. Each problem worth's 20 points.

1. Let M be a C^∞ manifold of dimension n.
   (a) Let W be an open subset of M. What does it mean for a function $f: W \to \mathbb{R}$ to be C^∞? How is $C^\infty(p)$ defined for any $p \in M$?
   (b) Give the definition of the tangent space $T_p(M)$ to M at point $p \in M$.
   (c) Prove that for any $p \in M$, $T_p(M)$ is an n dimensional real vector space.

2. Let M and N be C^∞ manifolds with M compact and let $f: M \to N$ be a C^∞ mapping. Let $q \in N$. Suppose that for every $p \in f^{-1}(q)$, $f_*: T_p(M) \to T_q(N)$ is an isomorphism. Show that $f^{-1}(q)$ consists of finitely many points of M.

3. Let M and N be C^∞ manifolds and $F: M \to N$ a C^∞ mapping.
   (a) Show that if $\forall p \in M$, $F_*: T_p(M) \to T_{F(p)}(N)$ is an isomorphism, then F is an open mapping.
   (b) Show that if M is compact and N is connected, then F is surjective.

4. Let X be a set and G a group acting transitively on X. Prove that for any $x, y \in X$, the isotropy subgroups G_x and G_y are conjugate subgroups of G.

5. Let G be a Lie group and let $L_g : G \to G$ be defined by $L_g : a \mapsto ag$ $\forall a \in G$. Call a vector field X on G left-invariant if for every $a \in G$ and every $g \in G$ $(L_g)_*(X(a)) = X(ag)$. Prove that there is a one-to-one correspondence between the set of all left-invariant vector fields on G and $T_e(G)$, where e is the identity of G.

6. Let M be a C^∞ mfd. Show that in general the bracket product $[\,,\,]$ on $\mathfrak{X}(M)$ is not bilinear over $C^\infty(M)$ by showing that $\forall X, Y \in \mathfrak{X}(M)$ and $\forall f, g \in C^\infty(M)$, $[fX, gY] = (f \cdot g)[X, Y] + (f \cdot (Xg))Y - (g \cdot (Yf))X$.

7. Prove the following C^∞ version of the Urysohn Lemma: If M is a C^∞ manifold and F_0 and F_1 are disjoint closed subsets of M, then there is a C^∞ function $f : M \to [0, 1]$ such that $f(F_0) \equiv 0$ and $f(F_1) \equiv 1$.

Ph.D. Exam in Differential Geometry

THERE ARE 4 PARTS
A, B, C, D

August 1991

A. Do any three

A1. Show that the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = (e^x \cos y, e^x \sin y)$$

is a local diffeomorphism, but not a diffeomorphism.

A2. Let $g: \mathbb{R}^3 \to \mathbb{R}$ be $g(x,y,z) = x^3 + 3y^3 + z^3 + 6x^2 y$. Show $g^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^3$.

A3. Let $X = (2x+y)\frac{\partial}{\partial x} + (x^2-y)\frac{\partial}{\partial y}$ and $Y = xy^3\frac{\partial}{\partial x} + x^3y^2\frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$. Calculate $[X,Y]$.

A4. Define $f: \mathbb{R}^2 \to \mathbb{R}^3$ by $f(x_1,x_2) = (x_1^2, 2x_1x_2, x_2^2) = (y_1, y_2, y_3)$. Find $f_*\left(\frac{\partial}{\partial x_2}\Big|_{(x_1,x_2)}\right)$ and $f^*(y_1 y_2 \, dy_3)$.

B. Do any two

B1. Show that the composition of two immersions is an immersion.

B2. Let $f: M \to N$ be a smooth mapping between two smooth manifolds of the same dimension. Assume M is compact. Show there exists $q \in N$ such that $f^{-1}(q)$ is a finite set.

B3. Let $f: M \to N$ be a smooth submersion (of rank = dim N). If M is compact and N is connected, prove that f is onto.

C. Do any two

C1. Let M be a compact oriented n-dimensional manifold. If η is a $(n-1)$-form on M prove that $d\eta$ vanishes at some point of M. Show by example that compactness is necessary.

$\int_M d\eta = \int_{\partial M} \eta = 0$

$H^n(M)$

C2. Let ω be a closed 1-form on M. Suppose that $\int_c \omega$ is an integer for every loop c in M. Show there exists a smooth mapping $f : M \to S^1 = \mathbb{R}/\mathbb{Z}$ such that $f^*(dt) = \omega$. (dt is the 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$ that comes from the coordinate t on $\mathbb{R}$.)

C3. Define Riemannian metric. Show every smooth manifold admits a Riemannian metric.

C4. Let $\pi : P \to M$ be a smooth principal G-bundle over M. Let G act on F on the left. Let $E = P \times_G F$ be the associated fiber bundle with fiber F. Show that the set of sections of E are in one-to-one correspondence with the set of maps $f : P \to F$ satisfying $f(ug) = g^{-1} \cdot f(u)$ for all $u \in P$ and $g \in G$.

D. Do this problem

Let G be the set of 2×2 real matrices

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & \frac{1}{x} \end{bmatrix} : x, y \in \mathbb{R},\ x \neq 0 \right\}$$

Regard (x,y) as a globally defined local coordinate system on G. Thus G is diffeomorphic to

$$\{ (x,y) \in \mathbb{R}^2 : x \neq 0 \}$$

~~(i)~~ Show G is a Lie group under matrix multiplica

(ii) Exhibit the Lie algebra $\mathfrak{g}$ of G as a sub-Lie algebra of $\mathfrak{gl}(2,\mathbb{R})$.

(iii) Find an explicit formula for the exponential map $\exp: \mathfrak{g} \to G$.

~~(iv)~~ Find a basis for the left-invariant one-forms on G, expressed in terms of the local coordinates.

MATH 641 FINAL EXAM

1) TENSORS OR NO

We have a Riemannian manifold M (with Levi-Civita covariant derivative ∇). We also have two fixed vector fields on M, Z and W, a fixed covector field θ, and a fixed real-valued function ϕ on M. We are going to define an operation $A : \mathcal{X}(M) \to \mathcal{X}(M)$. Your job is to determine whether or not A is tensorial, for each of these possible definitions of A:

a) $A(X) = [X, Z] \otimes W$
b) $A(X) = (\nabla_X Z) \otimes W$
c) $A(X) = (\nabla_Z X) \otimes W$
d) $A(X) = (X\phi)W$
e) $A(X) = (\theta X)W$
f) $A(X) = (X\phi)X$
g) $A(X) = (\nabla_X Z) \otimes X$
h) $A(X) = |X|W \quad (|X| = \langle X, X\rangle^{1/2})$

2) LIE DERIVATIVE OF COVECTORS

For any vector field V, $\mathcal{L}_V : \mathcal{X}(M) \to \mathcal{X}(M)$ is defined by $\mathcal{L}_V(X) = [V, X]$ (this is Lie derivative in direction V). We can extend to Lie derivative of covector fields by this definition for the covector field α:

$$\mathcal{L}_V(\alpha) : X \mapsto V(\alpha(X)) - \alpha(\mathcal{L}_V(X))$$

a) Show that the definition above yields that $\mathcal{L}_V(\alpha)$ really is a covector.
b) Show that for any $\phi \in \mathcal{F}(M)$, $\mathcal{L}_V(d\phi) = d(V\phi)$.

3) Connection, Geodesics, and Curvature in Product Manifolds

We have two Riemannian manifolds, (M_1, g^1) and (M_2, g^2). Let (M, g) be the metric product, i.e., $M = M_1 \times M_2$ and $g = \pi_1^* g^1 + \pi_2^* g^2$, where $\pi_i : M \to M_i$ is projection.

Perhaps more simply: For $X \in T_pM = T_{p_1}M_1 \times T_{p_2}M_2$ (where $p = (p_1, p_2)$), let X_1 and X_2 be the respective components of X in $T_{p_1}M_1$ and $T_{p_2}M_2$, so $X = (X_1, X_2)$ (i.e., $p_i = \pi_i p$ and $X_i = \pi_{i*}X$). Then with $g = \langle\ ,\ \rangle$ and $g^i = \langle\ ,\ \rangle_i$,

$$\langle X, Y\rangle = \langle X_1, Y_1\rangle_1 + \langle X_2, Y_2\rangle_2.$$

Let ∇ be the Levi-Civita connection for g in M and ∇_i the Levi-Civita connection for g_i in M_i.

a) Show that $\nabla_X Y = (\nabla^1_{X_1} Y_1, \nabla^2_{X_2} Y_2)$.

b) Show that a curve $c = (c_1, c_2)$ in M is a geodesic iff c_1 and c_2 are geodesics. $(c_i = \pi_i \circ c)$

c) Find a formula for the Riemann curvature tensor R in M in terms of the corresponding tensors R^i in M_i.

4) Poincaré Halfplane

$dx \otimes dx$ $dy \otimes dy$

Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and let $g = y^{-2}(dx^2 + dy^2)$. Show that the sectional curvature in M obeys $K = -1$.

(N.B.: $dx^2 = dx \otimes dx$)

(Hint: I found it easier to work with $g = \lambda(y)^2(dx^2 + dy^2)$ and only afterwards put in $\lambda(y) = y^{-1}$.)